

# COMPUTING THE CACTUS RANK OF A GENERAL FORM

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**ABSTRACT.** We present an approach to computing the cactus rank for a general homogeneous polynomial  $F \in K[x_0, \dots, x_n]$  and use it to show that the cactus rank of a generic cubic form is 12 (resp. 15, 18, 20, 22) when  $n = 6$ , (resp.  $n = 7, 8, 9, 10$ ).

## INTRODUCTION

The rank of a homogeneous polynomial or form is the minimal number of summands in a presentation of the form as a sum of powers of linear forms. In terms of apolarity the rank is the minimal degree of a smooth finite apolar subscheme, i.e. a subscheme whose homogeneous ideal is contained in the annihilator of the form in the ring of differential operators. Alexander and Hirschowitz computed the rank for a general form, in terms of its degree and number of variables. The minimal degree of a finite apolar subscheme for a form, allowing also singularities, is called the *cactus rank* and is, even for the general form, smaller than the rank. In fact for a general cubic form  $F \in K[x_0, \dots, x_n]$  the cactus rank is at most  $2n + 2$ , [Bernardi, Ranestad 2012, Theorems 3,4], and it is conjectured that this bound is sharp for  $n \geq 8$ .

In this paper we give an approach to computing the cactus rank for a general form, and we apply this technique for a general cubic form with  $6 \leq n \leq 10$ .

**Theorem 1.** *The cactus rank of a generic homogeneous cubic polynomial  $F \in K[x_0, \dots, x_n]$  is 12 (resp. 15, 18, 20, 22) when  $n = 6$ , (resp.  $n = 7, 8, 9, 10$ ).*

A minimal length apolar subscheme to a form is locally Gorenstein, so our approach is to estimate the dimension of the component of the Hilbert scheme that parameterizes finite local Gorenstein schemes. Now, any finite local Gorenstein scheme is the affine spectrum of an Artinian local Gorenstein quotient of a polynomial ring, and each such quotient is defined by the annihilator of a polynomial  $f$ , when you interpret the polynomial ring as the ring of differential operators on  $f$ . The length of the Artinian local Gorenstein quotient therefore coincides with the dimension of the space of partials of the polynomial  $f$ .

We estimate the dimension of the relevant component of the Hilbert scheme by estimating the dimension of the family of polynomials whose space of partials has a given dimension.

## 1. CACTUS RANK AND APOLAR GORENSTEIN SUBSCHEMES

Let  $S := K[x_0, \dots, x_n]$ , and consider the polynomial ring  $T := K[y_0, \dots, y_n]$  acting on  $S$  by differentiation. To be independent of characteristic, we let  $S$  be the divided power ring, so that the differentiation is contraction: If  $\alpha = (\alpha_0, \dots, \alpha_n)$  and  $\beta = (\beta_0, \dots, \beta_n)$  are

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multiindices, then

$$y^\alpha(x^{[\beta]}) = \begin{cases} x^{[\beta-\alpha]} & \text{if } \beta \geq \alpha, \\ 0 & \text{otherwise.} \end{cases}$$

In characteristic 0 we could have used ordinary differentiation, and therefore, by abuse of notation, we call  $g(f) \in S$  a partial of  $f \in S$  for any  $g \in T$ . Let  $S_1$  and  $T_1$  be the degree 1 parts of  $S$  and  $T$  respectively. With respect to the action above (classically known as *apolarity*),  $S_1$  and  $T_1$  are natural dual spaces and  $\langle x_0, \dots, x_n \rangle$  and  $\langle y_0, \dots, y_n \rangle$  are dual bases. In particular  $T$  is naturally the coordinate ring of  $\mathbb{P}(S_1)$ , the projective space of 1-dimensional subspaces of  $S_1$ , and vice versa. The annihilator of  $f \in S$  is an ideal in  $T$  which we denote by  $f^\perp \subset T$ . For a homogeneous polynomial  $F \in S$  the quotient  $T_F := T/F^\perp$  is graded Artinian and Gorenstein since  $F^\perp$  is homogeneous and  $T_F$  is finitely generated as a  $K$ -module (*Artinian*).  $T_F$  has a 1-dimensional socle, the annihilator of the unique maximal ideal (*Gorenstein*). The socle in  $T_F$  is the degree  $d$  part of the ring (see e.g. [Iarrobino, Kanev 1999, Lemma 2.14]).

**Definition 1.** A subscheme  $X \subset \mathbb{P}(S_1)$  is apolar to  $F \in S$  if its homogeneous ideal  $I_X \subset T$  is contained in  $F^\perp$ .

**Definition 2.** The cactus rank  $cr(F)$  of a form  $F \in S$  is the minimal length of an apolar subscheme  $X \subset \mathbb{P}(S_1)$  to  $F$ .

A general effective method to compute the cactus rank for a form seems out of reach at the moment. We restrict our attention to

**Question 1.** What is the cactus rank  $cr(F)$  of a general form  $F \in S$ ?

The strategy of the proof of Theorem 1 is to consider the Hilbert scheme  $Hilb_l(\mathbb{P}^n)$  of length  $l$  subschemes of  $\mathbb{P}^n$ . In the third Veronese embedding  $\mathbb{P}^n \rightarrow \mathbb{P}^{\binom{n+3}{3}-1}$ , the span of each subscheme  $\Gamma$  of length  $l$  is a linear space  $L_\Gamma$  of dimension at most  $l-1$ . The cactus rank of a general cubic form is then the minimal  $l$  such that these linear spaces fill  $\mathbb{P}^{\binom{n+3}{3}-1}$ . For this minimum it suffices to consider the subscheme  $Hilb_{G,l}(\mathbb{P}^n)$  of the Hilbert scheme parameterizing locally Gorenstein schemes, i.e. all of whose components are local Gorenstein schemes, cf. [Buczynska, Buczyński 2010, proof of Lemma 2.4]. So we let  $h_l$  be the dimension of the subscheme  $Hilb_{G,l}(\mathbb{P}^n)$ . Since the linear span of each length  $l$  subscheme is at most  $l-1$  the cactus rank of a general cubic form  $F \in K[x_1, \dots, x_n]$  is then at least the minimal  $l$  such that  $h_l + l - 1 \geq \binom{n+3}{3} - 1$ .

Casnati and Notari have shown that any local Gorenstein scheme of length at most 10 is smoothable (cf. [Casnati, Notari 2011]), so for our estimate we need only to consider locally Gorenstein schemes with a component  $\Gamma_0$  of length at least 11. Denote by  $Hilb_{G,l}^{loc} \mathbb{A}_0^n$  the subscheme of the Hilbert scheme parameterizing local Gorenstein schemes of length  $l$  supported at the origin of  $\mathbb{A}^n$ . Let

$$\gamma_l = \dim Hilb_{G,l}^{loc} \mathbb{A}_0^n.$$

Since any Gorenstein scheme of length at most 10 is smoothable, any Gorenstein scheme of length  $l \leq 21$  have at most one nonsmoothable component, say of length  $l' \leq l$ . Therefore we can compute  $h_l$  as

$$h_l = \max\{\gamma_{l'} + n + n(l-l') \mid 1 \leq l' \leq l\}.$$

Now each local Gorenstein scheme in  $\mathbb{A}^n$  supported at the origin is a Gorenstein scheme  $\Gamma(f) \subset \mathbb{A}^n$  defined by a polynomial  $f \in K[x_1, \dots, x_n]$  [Iarrobino 1994, Lemma 1.2]. In particular the affine coordinate ring of  $\Gamma(f)$  is the quotient  $K[y_1, \dots, y_n]/f^\perp$ . The length

of  $\Gamma(f)$  equals the dimension of  $\text{Diff}(f)$ , the space of partials of  $f$  of all orders. For our computations, we therefore consider the set of polynomials

$$V_l = \{f \in K[x_1, \dots, x_n] \mid \dim_K \text{Diff}(f) = l\} \subset K[x_1, \dots, x_n].$$

It is clearly a quasi affine algebraic set, when necessary we think of it as a scheme with its reduced scheme structure. Let  $f, g \in V_l$ , then

$$\Gamma(f) = \Gamma(g) \subset \mathbb{A}^n \Leftrightarrow \text{Diff}(f) = \text{Diff}(g).$$

But  $\text{Diff}(f) = \text{Diff}(g)$  if and only if  $\deg f = \deg g$  and  $g \in \text{Diff}(f)$ . Therefore

$$\gamma_l = \dim V_l - l.$$

Summarizing we get that the cactus rank of a general cubic form  $F \in K[x_0, \dots, x_n]$  with  $n \leq 10$  is at least the minimal  $l$  such that

$$h_l + l - 1 = \dim V_{l'} - l' + n + n(l - l') + l - 1 \geq \binom{n+3}{3} - 1,$$

i.e. when

$$\dim V_{l'} \geq \binom{n+3}{3} - n - (n+1)(l - l')$$

for  $l' \leq l$ .

In [Bernardi, Ranestad 2012] we showed that  $2n + 2$  is an upper bound for the cactus rank of a cubic in  $n + 1$  variables. The rank of a general cubic form  $F$  is given by the Alexander-Hirschowitz theorem (see [Alexander, Hirschowitz 1995]),

$$r(F) = \left\lceil \frac{1}{n+1} \binom{n+3}{3} \right\rceil,$$

when  $n \neq 4$ , and 8 when  $n = 4$ . Of course, the rank is an upper bound for the cactus rank. So for  $n \leq 7$  the cactus rank is strictly smaller than  $2n + 2$ . In fact, since Gorenstein schemes of length at most 10 are smoothable, the cactus rank of a general cubic form equals the rank for  $n \leq 5$ . Let

$$c(n) = \min \left\{ \left\lceil \frac{1}{n+1} \binom{n+3}{3} \right\rceil, 2n + 2 \right\}.$$

**Lemma 1.** *The cactus rank of a general cubic form in  $K[x_0, \dots, x_n]$  for  $6 \leq n \leq 10$ , is  $c(n)$ , if*

$$\dim V_l < \binom{n+3}{3} - n - (n+1)(c(n) - l - 1)$$

for  $11 < l \leq c(n) - 1$ .

*Proof.* It suffices to note that if  $l' \leq l < c(n)$ , then  $\binom{n+3}{3} - n - (n+1)(c(n) - 1 - l') \leq \binom{n+3}{3} - n - (n+1)(l - l')$ . Therefore

$$\dim V_l < \binom{n+3}{3} - n - (n+1)(c(n) - 1 - l) \quad \text{for } l < c(n)$$

means that the cactus rank of a general form is at least  $c(n)$ .  $\square$

## 2. HILBERT FUNCTIONS OF LOCAL ARTINIAN GORENSTEIN SCHEMES

In this section we interpret the Hilbert function of the associated graded ring of a local Artinian Gorenstein quotient of  $K[y_1, \dots, y_n]/f^\perp$  in terms of the polynomial  $f \in K[x_1, \dots, x_n]$ . In particular we recall and interpret Iarrobino's analysis of the Hilbert function and its symmetric decomposition. We will apply this analysis in the next section to estimate the dimension of the Hilbert scheme of local Gorenstein schemes with given Hilbert function.

Let  $f \in S = K[x_1, \dots, x_n]$  be a polynomial and let

$$f^\perp = \{g \in T \mid g(f) = 0\} \subset T$$

be the annihilator ideal with respect to differentiation. The local Artinian Gorenstein quotient ring  $T_f = T/f^\perp$  is naturally isomorphic to

$$\text{Diff}(f) = \{g(f) \mid g \in T\}$$

the space of all partials of  $f$ , as a  $T$ -module.

We consider Hilbert functions on graded rings associated to filtrations of  $T_f$ . The  $m$ -adic filtration

$$T_f = m^0 \supset m \supset m^2 \supset \dots \supset m^\delta \supset m^{\delta+1} = 0$$

where  $\delta = \deg f$ , defines an associated graded ring  $T_f^*$ . The L\"{o}ewy filtration

$$T_f = (0 : m^{\delta+1}) \supset (0 : m^\delta) \supset \dots \supset (0 : m^2) \supset (0 : m) \supset 0$$

is distinct from the  $m$ -adic filtration, but the successive quotients are dual:

$$m^i/m^{i+1} \cong ((0 : m^{\delta-i+1})/(0 : m^{\delta-i}))^*.$$

Therefore the Hilbert functions of the two associated graded rings are dual to each other.

Consider the following sequence of ideals of  $T_f^*$  induced by the Loewy filtration: For each  $a = 0, 1, 2, \dots$  let

$$C_a = \bigoplus_{i=0}^{\delta} C_{a,i} = \bigoplus_{i=0}^{\delta} ((0 : m^{\delta+1-a-i}) \cap m^i) / ((0 : m^{\delta+1-a-i}) \cap m^{i+1}) \subset T_f^*.$$

**Proposition 1.** [Iarrobino 1994, Theorem 1.5] *The quotients*

$$Q(a) = C_a/C_{a+1}, \quad a = 0, 1, 2, \dots$$

*are  $T_f^*$ -modules and satisfy the following reflexivity*

$$Q(a)_i^* \cong Q(a)_{\delta-a-i}.$$

*In particular, the Hilbert function  $\Delta_{a,f} = H(Q(a))$  is symmetric about  $(\delta - a)/2$ , and thus  $H(T_f^*)$  has a symmetric decomposition*

$$H(T_f^*) = \sum_a \Delta_{a,f}.$$

The possible symmetric decompositions of the Hilbert function is restricted by the fact that the partial sums of the symmetric decomposition are Hilbert functions of suitable quotients of  $T_f^*$ .

**Corollary 1.** [Iarrobino 1994, Section 5B, p. 69] *The Hilbert function of*

$$H(T_f^*/C_{\alpha+1}) = \sum_{a=0}^{\alpha} \Delta_{a,f}.$$

*In particular every partial sum  $\sum_{a=0}^{\alpha} \Delta_{a,f}$  is the Hilbert function of a  $K$ -algebra generated in degree 1.*

We now interpret the ideal  $C_a$  and the module  $Q(a)$  in terms of the space  $\text{Diff}(f)$  of partials of  $f$ . This interpretation depends on the isomorphism

$$\tau : T/f^\perp \rightarrow \text{Diff}(f), \quad g \mapsto g(f),$$

of  $T$ -modules and  $K$ -vector spaces. The image of  $(0 : m^i)$  under the map  $\tau$  is precisely  $\text{Diff}(f)_{i-1}$ , so the L ewy filtration

$$(0 : m) \subset (0 : m^2) \subset (0 : m^3) \subset \cdots \subset (0 : m^\delta) \subset (0 : m^{\delta+1}) = T/f^\perp$$

of  $T/f^\perp$  is mapped to the degree-filtration

$$K = \text{Diff}(f)_0 \subset \text{Diff}(f)_1 \subset \text{Diff}(f)_2 \subset \cdots \subset \text{Diff}(f)_\delta = \text{Diff}(f)$$

of  $\text{Diff}(f)$ , where  $\text{Diff}(f)_i$  is the subspace of partials of degree at most  $i$ .

Now  $(0 : m^i)/(0 : m^{i-1}) \cong (m^{i-1}/m^i)^*$ , so the integral function

$$H_f(0) = 1, H_f(i) = \dim_K \text{Diff}(f)_i - \dim_K \text{Diff}(f)_{i-1}, \quad i = 1, \dots, \delta,$$

coincides with the Hilbert function of  $T_f^*$  after reflection:

$$H_f(i) = H(T_f^*)(\delta - i).$$

On the other hand, the  $m$ -adic filtration

$$T/f^\perp \supset m \supset m^2 \supset \cdots \supset m^\delta \supset m^{\delta+1} = 0$$

corresponds to an order filtration on  $\text{Diff}(f)$ . The order  $\text{ord}(g)$  of  $g \in T$  is the smallest degree of a non-zero term of  $g$ . The order of a partial  $f'$  of  $f$  is the largest order of a  $g \in T$  such that  $f' = g(f)$ . Thus the image

$$\tau(m^i) \subset \text{Diff}_{\delta-i}(f)$$

is simply the space of partials of order at least  $i$  of  $f$ .

The isomorphism  $Q(a)_i^* \cong Q(a)_{\delta-a-i}$  allows us to interpret the vector space  $Q(a)_i^*$  as parameterizing partials of  $f$  of degree  $i$  and order  $\delta - a - i$ , modulo partials of lower degree or larger order.

More precisely, let  $\text{Diff}(f)_i^a \subset \text{Diff}(f)$  be the subspace of partials of degree at most  $i$  and order at least  $\delta - i - a$ , then

$$Q(a)_i^* \cong \text{Diff}(f)_i^a / (\text{Diff}(f)_{i-1}^a + \text{Diff}(f)_i^{a-1}).$$

So

$$\Delta_{a,f}(i) = \dim_K (Q(a)_i^*) = \dim_K (\text{Diff}(f)_i^a / (\text{Diff}(f)_{i-1}^a + \text{Diff}(f)_i^{a-1})).$$

Next, we enumerate possible Hilbert functions  $H$  and their possible symmetric decomposition

$$H = \sum_a \Delta_a.$$

We denote  $H$  by its values

$$H = (H(0), H(1), \dots, H(\delta))$$

and decomposition

$$H = \sum_a \Delta_a,$$

where each  $\Delta_a$  is symmetric around  $(\delta - a)/2$ , i.e.  $\Delta_a(i) = \Delta_a(\delta - a - i)$ . We let

$$H_\Sigma = 1 + H(1) + \cdots + H(\delta - 1) + 1$$

be the sum of the values.

By Corollary 1, both  $H$  and each partial sum

$$\Delta_{\leq \alpha} = \sum_{a=0}^{\alpha} \Delta_a$$

are Hilbert functions of  $K$ -algebras generated in degree 1, so there are some immediate restrictions on these functions. First, Hilbert functions  $H$  and  $\Delta_{\leq \alpha}$  have positive values for  $0 \leq i \leq \delta$  and satisfy the Macaulay growth condition (cf. [Macaulay 1927]): If the  $i$ -binomial expansion of  $\Delta_{\leq \alpha}(i)$  is

$$\Delta_{\leq \alpha}(i) = \binom{m_i}{i} + \binom{m_{i-1}}{i-1} + \cdots + \binom{m_j}{j}; \quad m_i > m_{i-1} > \cdots > m_j \geq j \geq 1,$$

then

$$(1) \quad \Delta_{\leq \alpha}(i+1) \leq \binom{m_i+1}{i+1} + \binom{m_{i-1}+1}{i} + \cdots + \binom{m_j+1}{j+1}.$$

**Example 1.** For  $H(1) = 8, H(2) \geq 5$  and  $H_{\Sigma} = 17$  the possible Hilbert functions  $H$  and their decompositions  $H = \sum_i \Delta_i$  that satisfy the Macaulay growth conditions are the following:

$$\begin{array}{lll} H & = & 1 \quad 8 \quad 7 \quad 1 \\ \Delta_0 & = & 1 \quad 7 \quad 7 \quad 1, \\ \Delta_1 & = & 0 \quad 1 \quad 0 \quad 0 \end{array}, \quad \begin{array}{lll} H & = & 1 \quad 8 \quad 6 \quad 1 \quad 1 \\ \Delta_0 & = & 1 \quad 1 \quad 1 \quad 1 \quad 1, \\ \Delta_1 & = & 0 \quad 5 \quad 5 \quad 0 \quad 0, \\ \Delta_2 & = & 0 \quad 2 \quad 0 \quad 0 \quad 0 \end{array}, \quad \begin{array}{lll} H & = & 1 \quad 8 \quad 5 \quad 1 \quad 1 \quad 1 \\ \Delta_0 & = & 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \\ \Delta_1 & = & 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0, \\ \Delta_2 & = & 0 \quad 4 \quad 4 \quad 0 \quad 0 \quad 0 \\ \Delta_3 & = & 0 \quad 3 \quad 0 \quad 0 \quad 0 \quad 0 \end{array}$$
  

$$\begin{array}{lll} H & = & 1 \quad 8 \quad 5 \quad 2 \quad 1 \\ \Delta_0 & = & 1 \quad 2 \quad 3 \quad 2 \quad 1 \\ \Delta_1 & = & 0 \quad 2 \quad 2 \quad 0 \quad 0, \\ \Delta_2 & = & 0 \quad 4 \quad 0 \quad 0 \quad 0 \end{array}, \quad \begin{array}{lll} H & = & 1 \quad 8 \quad 5 \quad 2 \quad 1 \\ \Delta_0 & = & 1 \quad 2 \quad 2 \quad 2 \quad 1 \\ \Delta_1 & = & 0 \quad 3 \quad 3 \quad 0 \quad 0, \\ \Delta_2 & = & 0 \quad 3 \quad 0 \quad 0 \quad 0 \end{array}.$$

In the Appendix we list all decompositions of Hilbert functions with  $H_{\Sigma} = 17$ .

### 3. POLYNOMIALS WITH GIVEN HILBERT FUNCTION

With the interpretation of the symmetric decomposition of the Hilbert function of  $T_f^*$  in terms of partials of  $f$ , we may now estimate the dimension of  $V_l$ . We consider the decomposition of  $V_l$  into subsets (and, when necessary, subschemes) according to the Hilbert function and its symmetric decomposition  $\Delta$ :

$$(2) \quad V_l = \cup_{H, \Delta} V_l^{H, \Delta} = \cup_H \{f \in V_l \mid H_f = H, \Delta_f = \Delta\}.$$

The union ranges over all possible Hilbert functions  $H$  whose total sum of values is  $l$  and all possible symmetric decompositions of  $H$ . Since the components  $V_l^H$  are disjoint, the dimension of  $V_l$  equals the dimension of the largest component  $V_l^{H, \Delta}$ . Therefore our aim is to compute an upper bound for the dimension of  $V_l^{H, \Delta}$  for a given  $H$  and  $\Delta$ .

Let  $f \in V_l^{H, \Delta}$  and let

$$f = f_{\delta} + f_{\delta-1} + \cdots + f_0$$

be the decomposition in homogeneous summands. We shall find restrictions on the summands  $f_i$  depending on the symmetric decomposition of the Hilbert function. The leading summand of  $f$  and of a partial  $g(f)$  is denoted  $\text{LS}(f)$  (resp.  $\text{LS}(g(f))$ ). In particular  $\text{LS}(f) = f_{\delta}$ . Notice that the Hilbert function  $H_f$  depends only on the first summands of  $f$ .

**Lemma 2.** Let  $H_f = \sum_{a \geq 0} \Delta_a$  be the symmetric decomposition of the Hilbert function  $H_f$  of  $f$ , and assume that  $\Delta_a = 0$  for  $a > \alpha$ . Then  $H_f = H_{f_\delta + f_{\delta-1} + \dots + f_{\delta-\alpha}}$ .

*Proof.* Observe that  $H_{f_\delta + f_{\delta-1} + \dots + f_{\delta-\alpha}} \geq \sum_{a=0}^\alpha \Delta_a = H_f$ . Since the other inequality is obvious, we get the equality.  $\square$

This lemma suggests a separate analysis of the higher degree and the lower degree summands of  $f$ . However, the leading summands of the partials depend on the variables involved in each homogeneous summand of  $f$ , so we find it more fruitful to consider these variables. The symmetric decomposition  $\Delta$  defines a filtration of  $S_1$  and  $T_1$ . Recall that each symmetric summand  $\Delta_{a,f}$  has values

$$\Delta_{a,f}(i) = \dim_K (\text{Diff}(f)_i^a / (\text{Diff}(f)_{i-1}^a + \text{Diff}(f)_i^{a-1})), \quad i = 1, \dots, \delta - a.$$

For  $i = 1$  we get

$$\Delta_{a,f}(1) = \dim_K (\text{Diff}(f)_1^a / (\text{Diff}(f)_0^a + \text{Diff}(f)_1^{a-1})),$$

so modulo the constants,  $\Delta_{a,f}(1)$  computes the dimensions of the quotients in the order filtration of partials of degree 1:

$$\text{Diff}(f)_1^0 \subset \text{Diff}(f)_1^1 \subset \dots \subset \text{Diff}(f)_1^{\delta-2} = \text{Diff}(f)_1.$$

Let  $f \in K[x_1, \dots, x_n]$  be a polynomial of degree  $\delta$  with symmetric decomposition  $\Delta_f = \Delta$  for its Hilbert function  $H_f$ . Let

$$b_i = \sum_{j=0}^i \Delta_j(1),$$

then, after a change of variables if necessary, we may assume that

$$(3) \quad \text{Diff}(f)_1^0 = \langle 1, x_1, \dots, x_{b_0} \rangle \subset \dots \subset \text{Diff}(f)_1^{\delta-2} = \langle 1, x_1, \dots, x_{b_{\delta-2}} \rangle = \text{Diff}(f)_1.$$

So for each  $i$ , the variables  $x_1, \dots, x_{b_i}$  span the partials of  $f$  of degree one and order at least  $\delta - 1 - i$ . In particular,  $\text{Diff}(f)_1 = \langle 1, x_1, \dots, x_{b_{\delta-2}} \rangle$ .

Let

$$\langle y_1, \dots, y_{b_0}, y_{b_0+1}, \dots, y_{b_1}, \dots, y_{b_{\delta-2}+1}, \dots, y_n \rangle = T_1$$

be the dual basis, i.e.  $y_i(x_j) = \delta_i^j$ . In terms of the isomorphism

$$\tau : T/f^\perp \mapsto \text{Diff}(f), \quad [p] \mapsto p(f),$$

a linear form  $x \in \langle x_{b_{a-1}+1}, \dots, x_{b_a} \rangle$  is the image of a polynomial i.e.  $x = p(f)$ , where  $[p] \in (0 : m^2) \cap m^{\delta-1-a}$ , so  $p \in T$  has order  $\delta - 1 - a$ . Furthermore in the dual basis of  $T_1$  there is a  $y \in T_1$  such that  $yp(f) = 1$ .

The variables  $x_1, \dots, x_{b_{\delta-2}}$  are precisely the ones that appear in the leading summands of partials of  $f$ .

**Lemma 3.** The leading summand of a partial of  $f$  of degree  $\delta - i$  and order  $j$  lies in  $K[x_1, \dots, x_{b_{i-j}}]$ .

*Proof.* Let  $g$  be the leading summand of a partial of  $f$  of degree  $\delta - i$  and order  $j$ , then any partial  $x$  of degree one of  $g$  is a partial of order at least  $\delta - i + j - 1$  of  $f$  and therefore lies in  $\langle x_1, \dots, x_{b_{i-j}} \rangle$ . Therefore  $g \in K[x_1, \dots, x_{b_{i-j}}]$ .  $\square$

There may be variables appearing in  $f$  that do not show up in the leading summands of partials of  $f$ . It is tempting to call them exotic variables, but we reserve exotic for the following related summands of  $f$ .

**Definition 3.** Let  $f = f_\delta + \dots + f_0 \in S$  be the homogeneous decomposition of  $f$ . An *exotic summand* of degree  $\delta - i$  is a form  $f_{\delta-i,\infty} \in \langle x_{b_i+1}, \dots, x_n \rangle K[x_1, \dots, x_n]$  such that the degree  $\delta - i$  homogeneous summand of  $f$  can be written as

$$f_{\delta-i} = f_{\delta-i,i} + f_{\delta-i,\infty},$$

with  $f_{\delta-i,i} \in K[x_1, \dots, x_{b_i}]$ .

Let  $f \in K[x_1, \dots, x_k]$  with  $\text{Diff}(f)_1 = \langle 1, x_1, \dots, x_k \rangle$  and let

$$(4) \quad W_f^m := \{\tilde{f} \in K[x_1, \dots, x_{k+m}] \mid \tilde{f} - f \in (x_{k+1}, \dots, x_{k+m}), \\ H_{\tilde{f}} = H_f, \text{Diff}(\tilde{f})_1 = \text{Diff}(f)_1\}.$$

For our dimension estimates we aim at a characterization of the family  $W_f^m$ . For this we choose a suitable basis for  $\text{Diff}(f)$ .

**Remark 1.** We choose a basis  $\{h_{i,j,r}\}_{(i,j,r) \in I}$  for  $\text{Diff}(f)$ , where

$$I = \{(i, j, r) \mid 0 \leq i \leq \delta, 0 \leq j \leq \delta - 2, 1 \leq r \leq \Delta_{f,j}(i)\},$$

such that for  $0 \leq i \leq \delta$  and  $0 \leq j \leq \delta - 2$ ,

$$h_{i,j,1}, \dots, h_{i,j,\Delta_{f,j}(i)}$$

are partials of  $f$  of degree  $i$  and order  $\delta - i - j$ . In particular, we may take  $h_{0,0,1} = 1$ , and  $h_{\delta,0,1} = f$ . We can then choose a set of polynomials  $\{g_{i,j,r}\}_{(i,j,r) \in I}$  in  $K[y_1, \dots, y_k]$  such that, for each  $(i, j, r) \in I$ ,

$$g_{i,j,r}(f) = h_{i,j,r} \text{ and } \text{ord}(g_{i,j,r}) = \text{ord}(h_{i,j,r}).$$

**Example 2.** We choose a basis for  $\text{Diff}(f)$  in an example. Let  $f = x_1^6 + x_1^3 x_2$ . Its Hilbert function decomposition is

$$\begin{array}{rcl} H_f & = & 1 \quad 2 \quad 1 \quad 1 \quad 1 \quad 1 \\ \Delta_{f,0} & = & 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \\ \Delta_{f,1}, \Delta_{f,2}, \Delta_{f,3} & = & 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \\ \Delta_{f,4} & = & 0 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0 \end{array}$$

and if we take

$$\begin{array}{llll} h_{6,0,1} = f, & h_{5,0,1} = x_1^5 + x_1^2 x_2, & h_{4,0,1} = x_1^4 + x_1 x_2, & h_{3,0,1} = x_1^3 + x_2, \\ h_{2,0,1} = x_1^2, & h_{1,0,1} = x_1, & h_{1,4,1} = x_2, & h_{0,0,1} = 1, \end{array}$$

we get a basis for  $\text{Diff}(f)$  as described above in Remark 1. Taking

$$g_{r,0,1} = y_1^{6-r}, \text{ for } 0 \leq r \leq 6 \text{ and } g_{1,4,1} = (y_1^3 - y_2),$$

we have elements in  $K[y_1, y_2]$  as described. Note that  $x_1^3 x_2$  is an exotic summand of  $f$ .

Now let  $\tilde{f} = x_1^6 + x_1^4 x_3 + x_1^3 x_2 + x_1^2 x_3^2 + x_1 x_2 x_3 + x_3^3$ . We can check that  $H_{\tilde{f}} = H_f$ ,  $\text{Diff}(\tilde{f})_1 = \text{Diff}(f)_1$  and

$$\tilde{f} = f + x_3 (x_1^4 + x_1^2 x_3 + x_1 x_2 + x_3^2),$$

therefore  $\tilde{f} \in W_f^1$ . Moreover  $\text{Diff}(\tilde{f})$  is spanned by the partials  $g_{6,0,1}(\tilde{f}), \dots, g_{0,0,1}(\tilde{f})$ . Since  $x_3$  does not appear in  $\text{Diff}(\tilde{f})_1$ , the summands  $x_1^4 x_3$ ,  $x_1^3 x_2 + x_1^2 x_3^2$  and  $x_1 x_2 x_3 + x_3^3$  are all exotic summands of  $\tilde{f}$ , in degrees 5, 4 and 3, respectively. Note that if we omit the summand  $x_1^4 x_3$ , we get a polynomial  $G = x_1^6 + x_1^3 x_2 + x_1^2 x_3^2 + x_1 x_2 x_3 + x_3^3$  with  $H_G \neq H_f$ , so we



get a polynomial that is no longer in  $W_f^1$ . Furthermore,  $x_1^2 x_3^2$  and  $x_1 x_2 x_3 + x_3^3$  are not exotic summands of  $G$ . The distinct property of  $\tilde{f}$ , as an element of  $W_f^1$ , is that we can write

$$\tilde{f} = f + x_3 \cdot g(f) + x_3^2 \cdot g^2(f) + x_3^3 \cdot g^3(f),$$

with  $g = g_{4,0,1}$ . Observe that the morphism

$$K[y_1, y_2, y_3]/\tilde{f}^\perp \rightarrow K[y_1, y_2]/f^\perp; \quad (y_1, y_2, y_3) \mapsto (y_1, y_2, g)$$

is an isomorphism, i.e.  $T_f$  and  $T_{\tilde{f}}$  are isomorphic rings.

**Example 3.** Let us look at a different case. Let  $f = x_1^7 + x_2^6 + x_1^2 x_2^2$ . Its Hilbert function decomposition is

$$\begin{array}{rcl} H_f & = & 1 \quad 2 \quad 3 \quad 2 \quad 2 \quad 2 \quad 1 \quad 1 \\ \Delta_{f,0} & = & 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \\ \Delta_{f,1} & = & 0 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 0 \quad 0 \\ \Delta_{f,2} & = & 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \\ \Delta_{f,3} & = & 0 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \end{array}$$

and if we take

$$\begin{array}{llll} h_{7,0,1} = f, & h_{6,0,1} = x_1^6 + x_1 x_2^2, & h_{5,0,1} = x_1^5 + x_2^2, & h_{5,1,1} = x_2^5 + x_1^2 x_2, \\ h_{4,0,1} = x_1^4, & h_{4,1,1} = x_2^4 + x_1^2, & h_{3,0,1} = x_1^3, & h_{3,1,1} = x_2^3, \\ h_{2,0,1} = x_1^2, & h_{2,1,1} = x_2^2, & h_{2,3,1} = x_1 x_2, & h_{1,0,1} = x_1, \\ h_{1,1,1} = x_2, & h_{0,0,1} = 1, & & \end{array}$$

we get a basis for  $\text{Diff}(f)$  as described in Remark 1. Taking

$$\begin{array}{llll} g_{7,0,1} = 1, & g_{6,0,1} = y_1, & g_{5,0,1} = y_1^2, & g_{5,1,1} = y_2, \\ g_{4,0,1} = y_1^3, & g_{4,1,1} = y_2^2, & g_{3,0,1} = y_1^4, & g_{3,1,1} = y_2^3, \\ g_{2,0,1} = y_1^5, & g_{2,1,1} = y_2^4, & g_{2,3,1} = y_1 y_2, & g_{1,0,1} = y_1^6, \\ g_{1,1,1} = y_2^5, & g_{0,0,1} = y_1^7, & & \end{array}$$

we have elements in  $K[y_1, y_2]$  as described.

Now let  $\tilde{f} = x_1^7 + x_1^5 x_3 + x_2^6 + x_1^2 x_2^2 + x_1 x_2 x_3 + x_2^2 x_3$ . Its Hilbert function is now different from  $H_f$  and has decomposition

$$\begin{array}{rcl} H_{\tilde{f}} & = & 1 \quad 3 \quad 4 \quad 3 \quad 3 \quad 2 \quad 1 \quad 1 \\ \Delta_{\tilde{f},0} & = & 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \\ \Delta_{\tilde{f},1} & = & 0 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 0 \quad 0 \\ \Delta_{\tilde{f},2} & = & 0 \quad 1 \quad 1 \quad 1 \quad 1 \quad 0 \quad 0 \quad 0 \\ \Delta_{\tilde{f},3} & = & 0 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \end{array}$$

Therefore  $\text{Diff}(\tilde{f})_1 = \text{Diff}(f)_1 \oplus \langle x_3 \rangle$ , and

$$\tilde{f} = f + x_3 (x_1^5 + x_1 x_2 + x_2^2).$$

We have a different situation to the one in previous example, but there are similarities: We may check that although  $\text{Diff}(\tilde{f})$  is not spanned by the partials  $g_{7,0,1}(\tilde{f}), \dots, g_{0,0,1}(\tilde{f})$ , these are linearly independent. Also

$$\begin{aligned} \tilde{f} &= x_1^7 + x_2^6 + x_1^2 x_2^2 + x_3 (x_1^5 + x_1 x_2 + x_2^2) + x_3^2 x_1^3 + x_3^3 x_1 \\ &= f + x_3 \cdot g(f) + x_3^2 \cdot g^2(f) + x_3^3 \cdot g^3(f) + G, \end{aligned}$$

where  $g = g_{5,0,1} + g_{2,3,1}$  and  $G = -x_3^2 x_1^3 - x_3^3 x_1$ . So if we set  $\tilde{f}' = \tilde{f} - G$ , we get that  $\tilde{f}' \in W_f^1$ , i.e.  $\tilde{f}'$  has the same Hilbert function as  $f$ .

These two examples suggest that a way to compute the dimension of  $V_l^{H,\Delta}$  is to deal separately with exotic summands and non exotic ones. Our goal is to write  $\dim(V_l^{H,\Delta})$  as a sum of the dimensions of two sets, one of which consists of polynomials without exotic summands (see Lemma 7), while the other consists of the possible exotic summands for a given polynomial. We will provide a bound for the dimension of the first set in Section 4. For the second set we first want to understand the exotic summands. In the remainder of this section we will show that they are intrinsically related to the partials of the polynomial (Corollary 3).

The following two results fully explain the exotic summands described in the previous two examples. Note that in those examples the monomials involving  $x_3$  that are present in the new exotic summands of  $\tilde{f}$  can all be obtained from sums of type  $\sum_{i \geq 0} x_3^i g^i(f)$ . We will see that if we have a polynomial  $f \in K[x_1, \dots, x_k]$  and add a few terms involving new variables  $x_{k+1}, \dots, x_{k+m}$  to get a polynomial  $\tilde{f} \in K[x_1, \dots, x_{k+m}]$  whose Hilbert function decomposition  $\Delta_{\tilde{f}}$  coincides with  $\Delta_f$  in the first summands, and ask for a few extra technical details, then the parts of the exotic summands of  $\tilde{f}$  involving the new variables  $x_{k+1}, \dots, x_{k+m}$  can all be written as sums of products of powers of these variables and partials of  $f$ .

**Lemma 4.** *Let  $f \in K[x_1, \dots, x_k]$  be a polynomial of degree  $\delta$  and  $a > 0$  an integer such that*

$$\text{Diff}(f)_1 = \langle 1, x_1, \dots, x_k \rangle, \text{ and } \Delta_{f,j}(1) = 0 \text{ for all } j \geq a.$$

*Consider a basis  $\{g_{i,j,r}(f) = h_{i,j,r}\}_{(i,j,r) \in I}$  for  $\text{Diff}(f)$ , with polynomials  $g_{i,j,r}$  in  $K[y_1, \dots, y_k]$ , as described in Remark 1.*

*Let  $\tilde{f} \in K[x_1, \dots, x_{k+m}]$  be a polynomial such that*

- (1)  $\tilde{f} - f \in (x_{k+1}, \dots, x_{k+m})$ ,
- (2)  $\langle x_1, \dots, x_k \rangle$  are the linear partials of  $\tilde{f}$  of order at least  $\delta - a$ .

*Then the partials  $\{g_{i,j,r}(\tilde{f})\}_{(i,j,r) \in I}$  are linearly independent elements of  $\text{Diff}(\tilde{f})$ . Moreover, if  $(i, j, r) \in I$  and  $j < a$ , then*

$$\deg g_{i,j,r}(\tilde{f}) = i \quad \text{and} \quad \text{ord } g_{i,j,r}(\tilde{f}) = \delta - i - j.$$

*and the leading summand satisfy  $\text{LS}(g_{i,j,r}(\tilde{f})) = \text{LS}(h_{i,j,r})$ .*

*Proof.* Write

$$\tilde{f} = f + x_{k+1}f_1 + \dots + x_{k+m}f_m.$$

Then, for each  $(i, j, r) \in I$ , since  $g_{i,j,r}$  is in  $K[y_1, \dots, y_k]$ , we have

$$g_{i,j,r}(\tilde{f}) = g_{i,j,r}(f) + x_{k+1}g_{i,j,r}(f_1) + \dots + x_{k+m}g_{i,j,r}(f_m).$$

The variables  $x_{k+1}, \dots, x_{k+m}$  do not occur in  $g_{i,j,r}(f)$ , so the leading summand  $\text{LS}(g_{i,j,r}(f))$  cannot be cancelled by  $x_{k+1}g_{i,j,r}(f_1) + \dots + x_{k+m}g_{i,j,r}(f_m)$ . The leading summands of the partials  $\{\text{LS}(g_{i,j,r}(\tilde{f}))\}_{(i,j,r) \in I}$  are linearly independent, since, if we order each of them lexicographically, the first monomials coincide with the first monomials of  $\{\text{LS}(g_{i,j,r}(f))\}_{(i,j,r) \in I}$ . Therefore the partials  $\{g_{i,j,r}(\tilde{f})\}_{(i,j,r) \in I}$  are linearly independent elements of  $\text{Diff}(\tilde{f})$ .

Now take  $j < a$ . We have

$$\deg g_{i,j,r}(\tilde{f}) \geq \deg g_{i,j,r}(f) = i.$$

On the other hand,  $\text{ord } g_{i,j,r}(\tilde{f}) \geq \text{ord } g_{i,j,r} = \text{ord } g_{i,j,r}(f) = \delta - i - j$ , so

$$\delta - \deg g_{i,j,r}(\tilde{f}) - \text{ord } g_{i,j,r}(\tilde{f}) \leq j < a.$$

Thus any partial of degree one of  $\text{LS}(g_{i,j,r}(\tilde{f}))$  has order at least

$$\deg g_{i,j,r}(f) + \text{ord } g_{i,j,r}(f) - 1 = \delta - j - 1 \geq \delta - a,$$

i.e. is contained in  $\langle x_1, \dots, x_k \rangle$ . Therefore  $\text{LS}(g_{i,j,r}(\tilde{f})) \in K[x_1, \dots, x_k]$ . In particular,

$$\text{LS}(g_{i,j,r}(\tilde{f})) = \text{LS}(g_{i,j,r}(f)) = \text{LS}(h_{i,j,r}),$$

and the partials  $\{g_{i,j,r}(\tilde{f})\}_{(i,j,r) \in I, j < a}$  satisfy

$$\deg g_{i,j,r}(\tilde{f}) = \deg g_{i,j,r}(f) = \deg h_{i,j,r} = i$$

and

$$\text{ord } g_{i,j,r}(\tilde{f}) = \text{ord } g_{i,j,r}(f) = \text{ord } h_{i,j,r} = \delta - i - j.$$

□

**Lemma 5.** *Let  $f \in K[x_1, \dots, x_k]$  and  $a > 0$  be as in Lemma 4. Let  $\{g_{i,j,r}(f) = h_{i,j,r}\}_{(i,j,r) \in I}$  be a basis for  $\text{Diff}(f)$  with  $g_{i,j,r} \in K[y_1, \dots, y_k]$ , as described in Remark 1.*

*Let  $\tilde{f} \in K[x_1, \dots, x_{k+m}]$  be a polynomial such that*

- (1)  $\tilde{f} - f \in \langle x_{k+1}, \dots, x_{k+m} \rangle$ ,
- (2) (a) *either*  $\text{Diff}(\tilde{f})_1 = \text{Diff}(f)_1 \oplus \langle x_{k+1}, \dots, x_{k+m} \rangle$ ,
- (b) *or*  $\text{Diff}(\tilde{f})_1 = \text{Diff}(f)_1$ ,
- (3)  $\langle x_1, \dots, x_k \rangle$  *are the linear partials of  $\tilde{f}$  of order at least  $\delta - a$ , and*
- (4) *for  $0 \leq j < a$ ,  $\Delta_{\tilde{f},j} = \Delta_{f,j}$ .*

*Then there are elements  $\phi_1, \dots, \phi_m \in \langle g_{i,j,r} \rangle_{(i,j,r) \in I}$  of order at least two such that*

$$(*) \quad \tilde{f} = \sum_{i_1, \dots, i_m \geq 0} x_{k+1}^{i_1} \cdots x_{k+m}^{i_m} \cdot (\phi_1^{i_1} \cdots \phi_m^{i_m})(f) + G,$$

*where  $G$  is a polynomial of degree at most  $\delta - a$ .*

*Proof.* By Lemma 4, the partials  $\{g_{i,j,r}(\tilde{f})\}_{(i,j,r) \in I}$  are linearly independent polynomials in  $\text{Diff}(\tilde{f})$ . Let  $\{g'_{i,j,r}\}_{(i,j,r) \in I'}$  be polynomials in  $K[y_1, \dots, y_{k+m}]$  such that

$$(5) \quad \text{Diff}(\tilde{f}) = \langle g_{i,j,r}(\tilde{f}) \rangle_{(i,j,r) \in I} \oplus \langle g'_{i,j,r}(\tilde{f}) \rangle_{(i,j,r) \in I'},$$

where  $g'_{i,j,r}(\tilde{f})$  have degree  $i \leq \delta - a - 1$  and order  $\delta - i - j$  and

$$I' = \{(i, j, r) \mid 0 \leq i \leq \delta - a - 1, a \leq j \leq \delta - 2, 1 \leq r \leq \Delta_{f,j}(i)\}.$$

We will use induction on  $m$ . Let  $m = 1$ , and write

$$\tilde{f} = f + x_{k+1}f_1.$$

Since  $x_{k+1}$  cannot occur in the leading summand of  $\tilde{f}$ , we have  $\deg f_1 \leq \delta - 2$ . Furthermore

$$y_{k+1}(\tilde{f}) = f_1,$$

i.e.  $f_1$  is a partial of  $\tilde{f}$ . Therefore  $f_1 = \phi(\tilde{f})$  for some  $\phi$  of order at least 1. By (5), we can write  $\phi = \phi_1 + \psi_1$ , where  $\phi_1 \in \langle g_{i,j,r} \rangle_{(i,j,r) \in I}$  and  $\psi_1 \in \langle g'_{i,j,r} \rangle_{(i,j,r) \in I'}$ . Then  $\phi_1(x_{k+1}) = 0$  and  $\psi_1(\tilde{f})$  has degree at most  $\delta - a - 1$ . We get

$$\begin{aligned}
\tilde{f} &= f + x_{k+1} \cdot \phi(\tilde{f}) \\
&= f + x_{k+1} \cdot (\phi_1 + \psi_1)(\tilde{f}) \\
&= f + x_{k+1} \cdot \phi_1(\tilde{f}) + x_{k+1} \cdot \psi_1(\tilde{f}) \\
&= f + x_{k+1} \cdot \phi_1(\tilde{f}) + G_1, & \deg G_1 \leq \delta - a \\
&= f + x_{k+1} \cdot \phi_1(f + x_{k+1} \cdot \phi_1(\tilde{f}) + G_1) + G_1 \\
&= f + x_{k+1} \cdot \phi_1(f) + x_{k+1}^2 \cdot \phi_1^2(\tilde{f}) + G_2, & \deg G_2 \leq \delta - a
\end{aligned}$$

and iterating this further we get

$$\tilde{f} = \sum_{i \geq 0} x_{k+1}^i \cdot \phi_1^i(f) + G, \quad \deg G \leq \delta - a.$$

Suppose now that the result holds for  $m - 1$ . Write

$$\tilde{f} = f + x_{k+1}f_1 + \cdots + x_{k+m}f_m,$$

where for each  $r \in \{1, \dots, m\}$ ,  $f_r \in K[x_1, \dots, x_k, x_{k+1}, \dots, x_{k+r}]$ , and let

$$\tilde{f}' = \tilde{f} - x_{k+m}f_m = f + x_{k+1}f_1 + \cdots + x_{k+m-1}f_{m-1}.$$

Then the partials  $\{g_{i,j,r}(\tilde{f}')\}_{(i,j,r) \in I}$  are linearly independent elements of  $\text{Diff}(\tilde{f}')$ . Observe that  $\text{LS}(g_{i,j,r}(\tilde{f}'))(x_1, \dots, x_k, 0, \dots, 0) = \text{LS}(h_{i,j,r})$ . Now, by induction there are elements  $\phi_1, \dots, \phi_m \in \langle g_{i,j,r} \rangle_{(i,j,r) \in I}$  such that

$$\tilde{f}' = \sum_{i_1, \dots, i_{m-1} \geq 0} x_{k+1}^{i_1} \cdots x_{k+m-1}^{i_{m-1}} \cdot \phi_1^{i_1} \cdots \phi_{m-1}^{i_{m-1}}(f) + G',$$

where the sum is taken as in (\*) and  $G'$  has degree at most  $\delta - a$ . Again, since  $x_{k+m}$  cannot be in the leading summand of  $\tilde{f}$ , we must have  $\deg f_m \leq \delta - 2$ . Now  $y_{k+m}(\tilde{f}) = f_m$ , which means that  $f_m$  is a partial of  $\tilde{f}$ , i.e.  $f_m = \phi(\tilde{f})$  for some form  $\phi$  of order greater than 0. As before, write  $\phi = \phi_m + \psi_m$ , with  $\phi_m \in \langle g_{i,j,r} \rangle_{(i,j,r) \in I}$  and  $\psi_m \in \langle g'_{i,j,r} \rangle_{(i,j,r) \in I'}$ . We now get

$$\begin{aligned}
\tilde{f} &= \tilde{f}' + x_{k+m} \cdot \phi(\tilde{f}) \\
&= \tilde{f}' + x_{k+m} \cdot (\phi_m + \psi_m)(\tilde{f}) \\
&= \tilde{f}' + x_{k+m} \cdot \phi_m(\tilde{f}) + x_{k+m} \cdot \psi_m(\tilde{f}) \\
&= \sum_{i_m \geq 0} x_{k+m}^{i_m} \cdot \phi_m^{i_m}(\tilde{f}') + G'' \\
&= \sum_{i_1, \dots, i_m \geq 0} x_{k+1}^{i_1} \cdots x_{k+m}^{i_m} \cdot \phi_1^{i_1} \cdots \phi_m^{i_m}(f) + G,
\end{aligned}$$

where the sum runs again as in (\*) and  $G$  has degree at most  $\delta - a$ .

It remains to show that  $\phi_1, \dots, \phi_m$  have order at least two. Suppose that  $\phi_r$  has order one for some  $r \in \{1, \dots, m\}$ . Write  $\phi_r = L + Q$ , where  $L = a_1y_1 + \cdots + a_ky_k \neq 0$  and  $\text{ord } Q \geq 2$ . Let  $t \in \{1, \dots, k\}$  be such that  $a_t \neq 0$ . Since  $x_t$  is a partial of order at least  $\delta - a$ , there exists

$\eta \in K[y_1, \dots, y_k]$  such that  $\eta(f) = x_t$  and  $\text{ord } \eta = \text{ord } x_t \geq \delta - a$ . Therefore  $\eta(G)$  is a constant. Note that  $(\phi_u \eta)(f) = \phi_u(x_t)$  is also a constant, for all  $u \in \{1, \dots, m\}$ , and moreover,  $\phi_r(x_t) = a_t \neq 0$ . So

$$\eta(\tilde{f}) = \sum_{i_1, \dots, i_m \geq 0} x_{k+1}^{i_1} \cdots x_{k+m}^{i_m} \cdot \phi_1^{i_1} \cdots \phi_m^{i_m} \eta(f) + \eta(G) = x_t + a_t x_{k+r} + M,$$

where  $M = \sum_{u \neq r} x_{k+u} \phi_u(x_t) + \eta(G)$  and has degree at most one. In fact, if  $i_1 + \dots + i_m \geq 2$ , then  $\phi_1^{i_1} \cdots \phi_m^{i_m} \eta(f) = 0$ . Now combining hypotheses (3) and (4), we can see that all linear partials of  $\tilde{f}$  of order at least  $\delta - a$  lie in  $\langle x_1, \dots, x_k \rangle$  and we get a contradiction, because  $\eta(\tilde{f})$  is a linear partial of  $\tilde{f}$  of order at least  $\delta - a$  and  $\eta(\tilde{f}) \notin \langle x_1, \dots, x_k \rangle$ .

This completes the proof of the lemma.  $\square$

We may now give a characterization of the family  $W_f^m$  defined in (4) for  $f \in K[x_1, \dots, x_k]$ . Recall the definition:

$$W_f^m := \{\tilde{f} \in K[x_1, \dots, x_{k+m}] \mid \tilde{f} - f \in (x_{k+1}, \dots, x_{k+m}) \\ H_{\tilde{f}} = H_f, \text{Diff}(\tilde{f})_1 = \text{Diff}(f)_1 = \langle x_1, \dots, x_k \rangle\}.$$

**Lemma 6.** *Let  $f \in K[x_1, \dots, x_k]$  be as in Lemma 4 and let  $\tilde{f} \in K[x_1, \dots, x_{k+m}]$  be any polynomial. Then  $\tilde{f} \in W_f^m$  if and only if there are elements  $\phi_1, \dots, \phi_m \in K[y_1, \dots, y_k]$  of order at least two such that*

$$(6) \quad \tilde{f} = \sum_{i_1, \dots, i_m \geq 0} x_{k+1}^{i_1} \cdots x_{k+m}^{i_m} \cdot (\phi_1^{i_1} \cdots \phi_m^{i_m})(f).$$

*Proof.* If  $\tilde{f} \in W_f^m$ , the hypotheses of Lemma 4 are satisfied and we have the additional assumption that  $H_{\tilde{f}} = H_f$ . Then the partials  $\{g_{i,j,r}(\tilde{f})\}_{(i,j,r) \in I}$  are linearly independent elements of  $\text{Diff}(\tilde{f})$ , and since  $\dim_K \text{Diff}(\tilde{f}) = \dim_K \text{Diff}(f)$ , we get that these partials span  $\text{Diff}(\tilde{f})$ . We can follow the proof of Lemma 5 and see that we get  $G = 0$  in (\*), so  $\tilde{f}$  can be written as in (6).

For the converse, if  $\tilde{f}$  is as in (6), clearly  $\tilde{f} - f \in (x_{k+1}, \dots, x_{k+m})$ . By Remark 1, we can consider a basis  $h_1, \dots, h_p$  for  $\text{Diff}(f)$  and polynomials  $g_1, \dots, g_p$  in  $K[y_1, \dots, y_k]$ , with  $g_r(f) = h_r$ . For any  $r \in \{1, \dots, p\}$ , we get

$$g_r(\tilde{f}) = \sum_{i_1, \dots, i_m \geq 0} x_{k+1}^{i_1} \cdots x_{k+m}^{i_m} \cdot (\phi_1^{i_1} \cdots \phi_m^{i_m})(g_r(f)),$$

and since all  $\phi_1, \dots, \phi_m$  have order at least two, we get  $\text{LS}(g_r(f)) = \text{LS}(g_r(\tilde{f}))$ , so  $g_1(\tilde{f}), \dots, g_p(\tilde{f})$  are linearly independent elements of  $\text{Diff}(\tilde{f})$ . Thus  $H_{\tilde{f}}(i) \geq H_f(i)$  for each  $i$ .

Now let  $\varphi \in K[y_1, \dots, y_{k+m}]$  be any differential. We wish to show that the fact that  $\varphi(\tilde{f}) \in \langle g_1(\tilde{f}), \dots, g_p(\tilde{f}) \rangle$  will imply  $H_{\tilde{f}} = H_f$ . Let  $r \in \{1, \dots, m\}$  and observe that

$$y_{k+r}(\tilde{f}) = \phi_r(\tilde{f}).$$

Therefore

$$\varphi(y_1, \dots, y_{k+m})(\tilde{f}) = \varphi(y_1, \dots, y_k, \phi_1, \dots, \phi_m)(\tilde{f}),$$

so we can assume that  $\varphi \in K[y_1, \dots, y_k]$ . But then

$$\varphi(\tilde{f}) = \sum_{i_1, \dots, i_m \geq 0} x_{k+1}^{i_1} \cdots x_{k+m}^{i_m} \cdot (\phi_1^{i_1} \cdots \phi_m^{i_m})(\varphi(f)),$$

so since  $\varphi(f) \in \langle g_1(f), \dots, g_p(f) \rangle$ , we get that  $\varphi(\tilde{f}) \in \langle g_1(\tilde{f}), \dots, g_p(\tilde{f}) \rangle$  and thus  $H_{\tilde{f}} = H_f$ .

To conclude our result, let  $\eta_1, \dots, \eta_k \in K[y_1, \dots, y_k]$  be such that  $\eta_r(f) = x_r$ , for  $1 \leq r \leq k$ . So

$$\eta_r(\tilde{f}) = \sum_{i_1, \dots, i_m \geq 0} x_{k+1}^{i_1} \cdots x_{k+m}^{i_m} \cdot (\phi_1^{i_1} \cdots \phi_m^{i_m})(x_r) = x_r,$$

since all  $\phi_1, \dots, \phi_m$  have order at least two. Therefore  $\text{Diff}(f)_1 \subseteq \text{Diff}(\tilde{f})_1$  and since their dimensions agree, we get equality.  $\square$

Note, as in Example 2, that the apolar Gorenstein rings  $T_{\tilde{f}}$  with  $\tilde{f} \in W_f^m$  are isomorphic to  $T_f$  but define different embeddings in  $\mathbb{A}^n$ .

**Corollary 2.** *Let  $f \in K[x_1, \dots, x_n]$  be written as a sum  $f = f_{(0)} + f_{(1)} + \cdots + f_{(\delta-1)}$ , where*

$$\begin{aligned} f_{(0)} &\in K[x_1, \dots, x_{b_0}], \\ f_{(i)} &\in \langle x_{b_{i-1}+1}, \dots, x_{b_i} \rangle K[x_1, \dots, x_{b_i}], \quad \text{for } 0 < i < \delta - 1, \end{aligned}$$

and

$$f_{(\delta-1)} \in \langle x_{b_{\delta-2}+1}, \dots, x_n \rangle K[x_1, \dots, x_n]$$

and denote

$$f_{(\leq i)} = f_{(0)} + f_{(1)} + \cdots + f_{(i)}.$$

For each  $i = 0, \dots, \delta - 2$ , let  $g_{i,1}, \dots, g_{i,r_i}$  be elements in  $K[y_1, \dots, y_{b_i}]$  such that the partials

$$g_{i,1}(f_{(\leq i)}), \dots, g_{i,r_i}(f_{(\leq i)})$$

form a basis for  $\text{Diff}(f_{(\leq i)})$ , as described in Remark 1. Then it is possible to write

$$(7) \quad f_{(\leq i)} = \sum_{j_{b_{i-1}+1}, \dots, j_{b_i} \geq 0} x_{b_{i-1}+1}^{j_{b_{i-1}+1}} \cdots x_{b_i}^{j_{b_i}} \cdot \left( \phi_{b_{i-1}+1}^{j_{b_{i-1}+1}} \cdots \phi_{b_i}^{j_{b_i}} \right) (f_{(\leq i-1)}) + G_i,$$

for  $0 < i < \delta - 1$ , and

$$(8) \quad f_{(\leq \delta-1)} = \sum_{j_{b_{\delta-2}+1}, \dots, j_n \geq 0} x_{b_{\delta-2}+1}^{j_{b_{\delta-2}+1}} \cdots x_n^{j_n} \cdot \left( \phi_{b_{\delta-2}+1}^{j_{b_{\delta-2}+1}} \cdots \phi_n^{j_n} \right) (f_{(\leq \delta-2)}) + G_{\delta-1},$$

where  $\phi_{b_{i-1}+1}, \dots, \phi_{b_i} \in \langle g_{i-1,1}, \dots, g_{i-1,r_{i-1}} \rangle$  and  $\phi_{b_{\delta-2}+1}, \dots, \phi_n \in \langle g_{\delta-2,1}, \dots, g_{\delta-2,r_{\delta-2}} \rangle$  are elements of order at least two such that, for each  $k \in \{1, \dots, n\}$ ,

$$\phi_k(f_{(\leq i-1)}) \text{ is a partial of } f_{(\leq i-1)} \text{ of degree } e, \text{ with } \delta - i - 1 \leq e \leq \delta - 2,$$

and  $G_i$  has degree at most  $\delta - i$  and, for  $i < \delta - 1$ , it does not involve any exotic summand of  $f_{(\leq i)}$ .

*Proof.* If we apply Lemma 5 to each pair  $(f_{(\leq i-1)}, f_{(\leq i)})$ , for  $0 < i \leq \delta - 1$ , we see that it is possible to write  $f_{(\leq i)}$  as in (7) with  $G_i$  a polynomial in  $K[x_1, \dots, x_{b_i}]$ , or, in case  $i = \delta - 1$ , as in (8), with  $\deg G_i \leq \delta - i$ . Since, by definition, an exotic summand of degree at most  $\delta - i$  must belong to  $\langle x_{b_i+1}, \dots, x_n \rangle K[x_1, \dots, x_n]$ , we get that  $G_i$  has no exotic summands of  $f_{(\leq i)}$ , for  $i < \delta - 1$ .  $\square$

**Corollary 3.** *Let  $f \in K[x_1, \dots, x_n]$ . Then we can write  $f = f_\alpha + f_\beta$  such that  $f_\alpha$  has no exotic summands and*

$$(9) \quad f_\beta = \sum_{i=1}^{\delta-1} \sum_{j_{b_{i-1}+1}, \dots, j_{b_i} \geq 0} x_{b_{i-1}+1}^{j_{b_{i-1}+1}} \dots x_{b_i}^{j_{b_i}} \cdot \left( \phi_{b_{i-1}+1}^{j_{b_{i-1}+1}} \dots \phi_{b_i}^{j_{b_i}} \right) (f_{(\leq i-1)}),$$

where, for each  $k \in \{1, \dots, n\}$ ,  $\phi_k$  has order at least 2 and

$$\phi_k (f_{(\leq i-1)}) \text{ is a partial of } f_{(\leq i-1)} \text{ of degree } e, \text{ with } \delta - i - 1 \leq e \leq \delta - 2.$$

as in Corollary 2.

*Proof.* If there exists  $i \leq \delta - 2$  such that  $f \in K[x_1, \dots, x_{b_i}]$ , then  $f = f_{(\leq i)}$  in the notation above. We apply (7) in Corollary 2 recursively to obtain  $f = f_\beta + G_i$ , with  $f_\beta$  as in (9). Then  $G_i$  does not involve exotic terms of  $f$ , so we can set  $f_\alpha = G_i$  and we are done.

Otherwise  $f = f_{(\leq \delta-1)}$ . We apply (7) in Corollary 2 recursively and obtain  $f = f_\beta + G_{\delta-1}$ , again with  $f_\beta$  as in (9). In this case  $G_{\delta-1} = a_1 x_1 + \dots + a_n x_n + b_0$  may involve exotic summands of  $f$ . However, we can choose a differential  $\eta$  such that  $\eta(f_{(\leq \delta-2)}) = 1$ . We may replace in (9), for  $b_{\delta-2} < r \leq n$ , each  $\phi_r$  by  $\phi_r + a_r \eta$ . Then we get  $f = f_\alpha + f_\beta$ , with  $f_\alpha$  a linear polynomial in  $K[x_1, \dots, x_{b_{\delta-2}}]$ , without exotic summands of  $f$ .  $\square$

With the decomposition of a polynomial of Corollary 3 we may in the next section estimate the dimension of the family of polynomials with a given Hilbert function.

#### 4. DIMENSION ESTIMATES

We estimate the dimensions that allow us to conclude the proof of Theorem 1. We first consider the family of possible exotic summands in a polynomial.

Let

$$V_l^{H, \Delta, \text{ne}} := \left\{ f \in V_l^{H, \Delta} \mid f \text{ has no exotic summands} \right\}.$$

**Lemma 7.** *The dimension  $d$  of the family  $V_l^{H, \Delta}$  satisfies*

$$d = \dim V_l^{H, \Delta, \text{ne}} + d_\infty,$$

where

$$(10) \quad d_\infty = \sum_{i=1}^{\delta-2} (n - b_i) \sum_{j=0}^{i-1} \Delta_j (\delta - i - 1) + (n - b_{\delta-2}).$$

*Proof.* By Corollary 3 the exotic summand of  $f_{(\leq i)}$  involving the variables  $x_{b_{i-1}+1}, \dots, x_{b_i}$  appears in

$$f_\beta = \sum_{i=1}^{\delta-2} \sum_{j_{b_{i-1}+1}, \dots, j_{b_i} \geq 0} x_{b_{i-1}+1}^{j_{b_{i-1}+1}} \dots x_{b_i}^{j_{b_i}} \cdot \left( \phi_{b_{i-1}+1}^{j_{b_{i-1}+1}} \dots \phi_{b_i}^{j_{b_i}} \right) (f_{(\leq i-1)}),$$

so the dimension of the family of possible exotic summands of  $f_{(\leq i)}$  involving the variables  $x_{b_{i-1}+1}, \dots, x_{b_i}$  is

$$d_{i, \infty} = (b_i - b_{i-1}) \sum_{j=0}^{i-2} \sum_{k=0}^j \Delta_k (\delta - 2 - j).$$

Similarly, the dimension of the family  $W_{f(\delta-1)}^{n-b_{\delta-2}}$ , of possible exotic summands of  $f$  involving the variables  $x_{b_{\delta-2}+1}, \dots, x_n$ , is

$$d_{\delta-1,\infty} = (n - b_{\delta-2}) \sum_{j=0}^{\delta-3} \sum_{k=0}^j \Delta_k(\delta-2-j) + (n - b_{\delta-2}).$$

The last summand concerns linear forms in the variables  $x_{b_{\delta-2}+1}, \dots, x_n$ . These values add up to

$$d_{\infty} = \sum_{i=1}^{\delta-2} (b_i - b_{i-1}) \sum_{j=0}^{i-2} \sum_{k=0}^j \Delta_k(\delta-2-j) + (n - b_{\delta-2}) \sum_{j=0}^{\delta-3} \sum_{k=0}^j \Delta_k(\delta-2-j) + (n - b_{\delta-2}).$$

And carefully rearranging this sum, we can get the expression in (10).  $\square$

It remains to estimate the dimension of  $V_l^{H,\Delta,\text{ne}}$ . Thus we assume that  $f = f_{\alpha}$  and has no exotic summands and consider the decomposition of  $f$  in homogeneous summands:

$$f = f_{\delta} + \dots + f_1 + f_0.$$

We compute inductively, starting with  $f_{\delta}$ , an upper bound  $d_i$  for the dimension of the family of summands  $f_i$  for the family of polynomials  $f \in V_l^{H,\Delta,\text{ne}}$  with a given partial sum  $f_{\delta} + \dots + f_{i+1}$ .

In the largest degree  $\delta$  the summand  $f_{\delta}$  is a form with Hilbert function  $\Delta_0$ . In particular, it depends on  $b_0 = \Delta_0(1)$  variables, with a  $\Delta_0(2)$ -dimensional space of second order partials, so if  $\delta \geq 4$  we may take

$$(11) \quad d_{\delta} \leq \binom{b_0 - 1 + \delta}{b_0 - 1} - 1/2 \left[ \binom{b_0 + 1}{2} - \Delta_0(2) + 1 \right] \left[ \binom{b_0 + 1}{2} - \Delta_0(2) \right].$$

The second summand is a lower bound for the codimension of the rank  $\Delta_0(2)$  locus for a symmetric submatrix of a catalecticant matrix in degrees  $(\delta - 2) \times 2$ , therefore if  $\delta = 3$  we simply take  $d_3 \leq \binom{b_0+2}{b_0-1}$ .

For the lower degree summands in  $f$ , the dimension estimate is more involved. If  $i > 0$ , and  $\delta - i \geq 4$ , then the form  $f_{\delta-i}$  is a form of degree  $\delta - i$  in  $b_i$  variables. Let  $D_2(f_{\delta-i}) \subset T_2$  be a subspace of quadratic forms in  $y_1, \dots, y_{b_i}$ , isomorphic to the space of partials of  $f_{\delta-i}$  of order 2. Then  $\tau(D_2(f)) \subset \text{Diff}(f)$  is a subspace of partials of degree at least  $\delta - i - 2$  and order at least 2. Therefore the space of partials of order 2 of  $f_{\delta-i}$  is bounded by

$$c_i = \sum_{j=0}^i \sum_{k=2}^{i-j+2} \Delta_j(\delta - j - k)$$

so it depends on

$$(12) \quad d_{\delta-i} = \binom{\delta - i + b_i - 1}{b_i - 1} - \binom{\binom{b_i+1}{2} - c_i + 1}{2}$$

parameters.

In degrees three, two, one and zero the estimates are

$$(13) \quad d_3 \leq \binom{b_{\delta-3} + 2}{3}, \quad d_2 \leq \binom{b_{\delta-2} + 1}{2}, \quad d_1 = b_{\delta-2}, \quad \text{and } d_0 = 1.$$

Finally, having estimated the dimension of the space of exotic summands, and of the polynomials with a given Hilbert function, with a given filtration of space of variables, it remains to estimate the dimension of the set of filtrations of  $\langle x_1, \dots, x_n \rangle$  with respect to the



symmetric decomposition of  $H$ . It is parameterized by the flag variety  $Fl(b_0, b_1, \dots, H(1), n)$ . The dimension of  $Fl(b_0, b_1, \dots, b_\delta, n)$  is

$$(14) \quad d_\Delta \leq b_0(n - b_0) + (b_1 - b_0)(n - b_1) + \dots + (b_\delta - b_{\delta-1})(n - b_\delta) = \sum_j \Delta_j(1)(n - b_j)$$

Putting formulas (10), (11), (12), (13) and (14) together, we get:

**Proposition 2.** *The dimension of the subscheme  $V_l^{H,\Delta}$  of polynomials  $f \in K[x_1, \dots, x_n]$  with  $\dim_K \text{Diff}(f) = l$ , Hilbert function  $H$  and symmetric decomposition  $\Delta$  is bounded by*

$$(15) \quad \dim V_l^{H,\Delta} \leq d = \sum_{i \geq 0} d_{\delta-i} + d_\Delta + d_\infty$$

$$= \binom{b_0 - 1 + \delta}{b_0 - 1} - 1/2 \left[ \binom{b_0 + 1}{2} - \Delta_0(2) + 1 \right] \left[ \binom{b_0 + 1}{2} - \Delta_0(2) \right]$$

$$+ \sum_{i=1}^{\delta-4} \left[ \binom{\delta - i + b_i - 1}{b_i - 1} - \binom{\binom{b_i + 1}{2} - c_i + 1}{2} \right]$$

$$+ \binom{b_{\delta-3} + 2}{3} + \binom{b_{\delta-2} + 1}{2} + b_{\delta-2} + 1$$

$$+ \sum_j \Delta_j(1)(n - b_j)$$

$$+ \sum_{i=1}^{\delta-2} (n - b_i) \left( \sum_{k=0}^{i-1} \Delta_k(\delta - i - 1) \right) + (n - b_{\delta-2}).$$

**Example 4.** With  $n = 8$  and

$$\begin{array}{rcl} H & = & 1 \quad 4 \quad 5 \quad 4 \quad 1 \quad 1 \quad 1 \\ \Delta_0 & = & 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \\ \Delta_2 & = & 0 \quad 3 \quad 4 \quad 3 \quad 0 \quad 0 \quad 0 \end{array}$$

we get that  $\delta = 6$ ,  $l = 17$ ,  $\Delta_0(1) = 1$ ,  $\Delta_2(1) = 3$  and

$$(b_0, b_1, b_2, b_3, b_4, b_5, b_6) = (1, 1, 4, 4, 4, 4, 4).$$

Therefore  $d_1 = 4$ ,  $d_2 = 10$ ,  $d_3 = 20$ ,  $d_4 = 29$ ,  $d_5 = 1$ ,  $d_6 = 1$ ,  $d_\Delta = 19$ ,  $d_\infty = 51$ , hence  $d = 1 + d_1 + \dots + d_6 + d_\Delta + d_\infty = 136$  which is less than  $\binom{8+3}{3} - 8 = 9 \cdot 17 + 4 = 157$ .

The dimension estimates for the other Hilbert functions are computed similarly for  $n = 6, 7, 8, 9, 10$  by using [Matlab 2011]. In the Appendix we list all of possible Hilbert functions and symmetric decompositions  $l = 17$  and  $n = 8$ . A similar list for  $n = 6, 7, 9, 10$  and  $l \leq 21$  is available at [Bernardi, Marques, Ranestad 2012].

We only get one estimate exceeding the bound  $\binom{n+3}{3} - n$ , for  $n = 10$ , namely, for the Hilbert function  $(1, 5, 4, 4, 5, 1, 1)$ , with decomposition  $(1, 1, 1, 1, 1, 1) + (0, 4, 3, 3, 4, 0, 0)$ . Our enumeration of Hilbert functions was however coarse:

**Lemma 8.** *There is no polynomial  $f$  with Hilbert function  $H_f = (1, 5, 4, 4, 5, 1, 1)$ .*

*Proof.* Assume that  $f$  is a polynomial with Hilbert function  $H_f = (1, 5, 4, 4, 5, 1, 1)$ . First we note that the symmetric decomposition of  $H_f$  must be  $(1, 1, 1, 1, 1, 1) + (0, 4, 3, 3, 4, 0, 0)$ . So, by Lemma 2, we may assume that  $f$  has summands only of degree 6 and 5, i.e.  $f = f_6 + f_5$ . The leading summand  $f_6$  of  $f$  has Hilbert function  $(1, 1, 1, 1, 1, 1)$ , so  $f_6$  must be a pure power, i.e.  $f_6 = x_1^6$ . Furthermore,  $H_f(1) = 5$ , so we may assume  $f = x_1^6 + f_5$  with  $f_5 \in K[x_1, \dots, x_5]$ .

Consider the Hilbert function  $H_{f_5}$  of  $f_5$ . It is symmetric and has  $H_{f_5}(1) \geq 4$  and  $H_{f_5}(2) \geq 3$ . If  $H_{f_5}(2) = H_{f_5}(3) = 3$ , then, by the Macaulay growth bound, cf. (1), we get  $4 \leq H_{f_5}(4) = H_{f_5}(1) \leq 3$ , which is a contradiction. Furthermore, if  $H_{f_5}(1) = 4$  then  $f_5 \in K[x_2, \dots, x_5]$  and so  $H_{f_5}(4) = 3$ , a contradiction. Therefore  $H_{f_5} = (1, 5, 4, 4, 5, 1)$ .

Set  $f_5 = x_1g + h$ , with  $g \in K[x_1, \dots, x_5]$  and  $h \in K[x_2, \dots, x_5]$ . We consider the Hilbert functions  $H_h$  and  $H_g$  of  $h$  and  $g$  respectively. Since at least one degree 3 partial of  $f$  must involve  $x_1^3$ , we have  $H_h(3) \leq 3$ , so it is, as above, bounded by  $(1, 3, 3, 3, 3, 1)$ . In particular  $h = 0$  or  $1 \leq H_h(1) \leq 3$ .

Since  $H_f(4) = 5$  and  $y_1(f)$  has degree 5, the partials  $y_2(f) = y_2(f_5), \dots, y_5(f) = y_5(f_5)$  are linearly independent and have order one. Thus

$$x_1^4, x_1y_2(g) + y_2(h), \dots, x_1y_5(g) + y_5(h)$$

are linearly independent.

If  $h = 0$ , then

$$\text{LS}(y_1^2(f)), y_2(f), \dots, y_5(f) = x_1^4, x_1y_2(g), \dots, x_1y_5(g)$$

are linearly independent leading summands of partials of degree 4. But then

$$\text{LS}(y_1^3(f)), y_1y_2(f), \dots, y_1y_5(f) = x_1^3, y_2(g), \dots, y_5(g)$$

are linearly independent leading summands of partials of degree 3, a contradiction since  $H_f(3) = 4$ .

If  $H_h(1) = 1$ , then we may assume that  $y_2(h) = y_3(h) = y_4(h) = 0$  and  $h = x_5^5$ . But then

$$x_1^4, x_1y_2(g), x_1y_3(g), x_1y_4(g), x_1y_5(g) + x_5^4$$

are linearly independent, and hence

$$x_1^3, y_2(g), y_3(g), y_4(g),$$

span the space of leading summands of partials of degree 3 of  $f$ . So  $y_5(g)$  belongs to  $\langle x_1^3, y_2(g), y_3(g), y_4(g) \rangle$ . By a change of variables, we may assume that  $y_5(g) = 0$ , and therefore that  $x_5^4$  is the leading summand of a partial of degree 4. But then  $x_5^3$  is the leading summand of a partial of degree 3, i.e.  $x_5^3 \in \langle y_2(g), y_3(g), y_4(g) \rangle$ , which contradicts the fact that  $y_5(g) = 0$ .

If  $H_h(1) = 2$ , then we may assume that  $y_2(h) = y_3(h) = 0$  and  $h = x_4^5 + x_5^5$  or  $h = x_4x_5^4$ , while  $y_5(g) \in \langle x_1^3, y_2(g), y_3(g), y_4(g) \rangle$ . By change of variables we may assume  $y_5(g) = 0$ , and therefore that  $x_5^4$  or  $x_4x_5^3$  is the leading summand of a partial of degree 4, i.e.  $x_5^3$  or  $x_4x_5^2$  is the leading summand of a partial of degree 3. Thus

$$x_1^4, x_1y_2(g), \dots, x_1y_4(g) + y_4(h), y_5(h)$$

are linearly independent. So

$$y_4(g) \in \langle x_1^3, y_2(g), y_3(g), y_5^2(h) \rangle.$$

But  $y_5(g) = 0$ , while  $y_5^3(h) \neq 0$ , so  $y_4(g) \in \langle x_1^3, y_2(g), y_3(g) \rangle$ . Therefore, by a change of variables, we may assume that also  $y_4(g) = 0$ , which means that the partials of  $h$  are independent of the partials of  $g$ , and hence  $H_f(3) \geq 1 + H_g(3) + H_h(3) = 5 > 4 = H_f(3)$ , a contradiction.

If  $H_h(1) = 3$ , then we may assume that  $y_2(h) = 0$ , while  $H_h = (1, 3, 3, 3, 3, 1)$ , and hence  $h$  have three independent partials  $\langle h_1, h_2, h_3 \rangle$  of degree 3.

In particular,

$$x_1^4, x_1y_2(g), x_1y_3(g) + y_3(h), x_1y_4(g) + y_4(h), x_1y_5(g) + y_5(h)$$

are linearly independent, while the space of cubic forms

$$\langle x_1^3, y_2(g), y_3(g), y_4(g), y_5(g), h_1, h_2, h_3 \rangle$$

have dimension at most 4. Thus  $y_2(g), y_3(g), y_4(g), y_5(g) \in \langle x_1^3, h_1, h_2, h_3 \rangle$ . Observe that  $y_2(g) \notin \langle x_1^3 \rangle$ , otherwise  $x_1^4$  and  $x_1 y_2(g)$  would not be independent, so there is a  $y_i$ , with  $i > 1$ , such that  $y_2 y_i(g) \neq 0$ . But  $y_2(x_1^3) = y_2(h_i) = 0$ , so  $y_i(g) \notin \langle x_1^3, h_1, h_2, h_3 \rangle$ , contradicting the above.  $\square$

The results of our computations, applying the Proposition 2 to all possible Hilbert functions  $H$  and symmetric decompositions  $\Delta$ , may therefore be summarized as follows:

**Proposition 3.** *The dimension of the subscheme  $V_l$  of polynomials  $f \in K[x_1, \dots, x_n]$  with  $\dim_K \text{Diff } f = l$  is bounded by*

$$\dim V_l < \binom{n+3}{3} - n - (n+1)(c(n) - l - 1)$$

when  $11 < l \leq c(n) - 1$  and  $6 \leq n \leq 10$ .

With Lemma 1 this concludes the proof of Theorem 1.

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## APPENDIX

Here we list all the possible Hilbert functions  $H$  and their possible symmetric decompositions for which  $n = 8$ ,  $H_\Sigma = 17$  and that satisfy the Macaulay growth condition (1). All the computations are done by using [Matlab 2011]. For all of them we will write the corresponding value of  $d = \sum_{i \geq 0} d_{\delta-i} + d_\Delta + d_\infty$  as computed in (15). For  $11 \leq H_\Sigma \leq 16$  the corresponding list of Hilbert functions can be found in [Iarrobino 1994] §5.F.

We explain here the compact notation that we will use: The Hilbert function decomposition

$$\begin{array}{rcccccccc} H & = & 1 & 4 & 5 & 4 & 1 & 1 & 1 \\ \Delta_0 & = & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \Delta_2 & = & 0 & 3 & 4 & 3 & 0 & 0 & 0 \end{array}$$

is written

$$(1, 4, 5, 4, 1, 1, 1) \rightarrow (1, 1, 1, 1, 1, 1, 1), (0, 3, 4, 3, 0, 0, 0).$$

The fact that  $(1, 1, 1, 1, 1, 1, 1)$  and  $(0, 3, 4, 3, 0, 0, 0)$  correspond to  $\Delta_0$  and  $\Delta_2$  respectively, is clear from the centre of symmetry ( $\Delta_a$  is symmetric about  $(\delta - a)/2$  as remarked in Proposition 1 following [Iarrobino 1994, Theorem 1.5]).

Here the list of all possible Hilbert functions with  $H_\Sigma = 17$ :

- $(1, 8, 7, 1) \rightarrow (1, 7, 7, 1), (0, 1, 0, 0), d = 136.$
- $(1, 4, 7, 4, 1) \rightarrow (1, 4, 7, 4, 1), d = 128.$
- $(1, 5, 5, 5, 1) \rightarrow (1, 5, 5, 5, 1), d = 119.$
- $(1, 5, 6, 4, 1) \rightarrow (1, 4, 6, 4, 1), (0, 1, 0, 0, 0), d = 124.$
- $(1, 5, 6, 4, 1) \rightarrow (1, 4, 5, 4, 1), (0, 1, 1, 0, 0), d = 128.$
- $(1, 5, 7, 3, 1) \rightarrow (1, 3, 6, 3, 1), (0, 1, 1, 0, 0), (0, 1, 0, 0, 0), d = 117.$
- $(1, 5, 7, 3, 1) \rightarrow (1, 3, 5, 3, 1), (0, 2, 2, 0, 0), d = 124.$
- $(1, 6, 4, 5, 1) \rightarrow (1, 5, 4, 5, 1), (0, 1, 0, 0, 0), d = 108.$
- $(1, 6, 5, 4, 1) \rightarrow (1, 4, 5, 4, 1), (0, 2, 0, 0, 0), d = 118.$
- $(1, 6, 5, 4, 1) \rightarrow (1, 4, 4, 4, 1), (0, 1, 1, 0, 0), (0, 1, 0, 0, 0), d = 122.$
- $(1, 6, 5, 4, 1) \rightarrow (1, 4, 3, 4, 1), (0, 2, 2, 0, 0), d = 131.$
- $(1, 6, 6, 3, 1) \rightarrow (1, 3, 6, 3, 1), (0, 3, 0, 0, 0), d = 112.$
- $(1, 6, 6, 3, 1) \rightarrow (1, 3, 5, 3, 1), (0, 1, 1, 0, 0), (0, 2, 0, 0, 0), d = 115.$
- $(1, 6, 6, 3, 1) \rightarrow (1, 3, 4, 3, 1), (0, 2, 2, 0, 0), (0, 1, 0, 0, 0), d = 122.$
- $(1, 6, 6, 3, 1) \rightarrow (1, 3, 3, 3, 1), (0, 3, 3, 0, 0), d = 134.$
- $(1, 6, 7, 2, 1) \rightarrow (1, 2, 3, 2, 1), (0, 4, 4, 0, 0), d = 129.$
- $(1, 7, 4, 4, 1) \rightarrow (1, 4, 4, 4, 1), (0, 3, 0, 0, 0), d = 110.$
- $(1, 7, 4, 4, 1) \rightarrow (1, 4, 3, 4, 1), (0, 1, 1, 0, 0), (0, 2, 0, 0, 0), d = 114.$
- $(1, 7, 5, 3, 1) \rightarrow (1, 3, 5, 3, 1), (0, 4, 0, 0, 0), d = 108.$
- $(1, 7, 5, 3, 1) \rightarrow (1, 3, 4, 3, 1), (0, 1, 1, 0, 0), (0, 3, 0, 0, 0), d = 111.$
- $(1, 7, 5, 3, 1) \rightarrow (1, 3, 3, 3, 1), (0, 2, 2, 0, 0), (0, 2, 0, 0, 0), d = 118.$
- $(1, 7, 6, 2, 1) \rightarrow (1, 2, 3, 2, 1), (0, 3, 3, 0, 0), (0, 2, 0, 0, 0), d = 114.$
- $(1, 7, 6, 2, 1) \rightarrow (1, 2, 2, 2, 1), (0, 4, 4, 0, 0), (0, 1, 0, 0, 0), d = 128.$
- $(1, 7, 7, 1, 1) \rightarrow (1, 1, 1, 1, 1), (0, 6, 6, 0, 0), d = 143.$
- $(1, 8, 3, 4, 1) \rightarrow (1, 4, 3, 4, 1), (0, 4, 0, 0, 0), d = 100.$
- $(1, 8, 4, 3, 1) \rightarrow (1, 3, 4, 3, 1), (0, 5, 0, 0, 0), d = 102.$
- $(1, 8, 4, 3, 1) \rightarrow (1, 3, 3, 3, 1), (0, 1, 1, 0, 0), (0, 4, 0, 0, 0), d = 105.$
- $(1, 8, 5, 2, 1) \rightarrow (1, 2, 3, 2, 1), (0, 2, 2, 0, 0), (0, 4, 0, 0, 0), d = 102.$
- $(1, 8, 5, 2, 1) \rightarrow (1, 2, 2, 2, 1), (0, 3, 3, 0, 0), (0, 3, 0, 0, 0), d = 111.$

$(1, 8, 6, 1, 1) \rightarrow (1, 1, 1, 1, 1), (0, 5, 5, 0, 0), (0, 2, 0, 0, 0), d = 121.$   
 $(1, 4, 4, 4, 3, 1) \rightarrow (1, 3, 4, 4, 3, 1), (0, 1, 0, 0, 0, 0), d = 133.$   
 $(1, 4, 4, 4, 3, 1) \rightarrow (1, 3, 3, 3, 3, 1), (0, 1, 1, 1, 0, 0), d = 146.$   
 $(1, 4, 4, 5, 2, 1) \rightarrow (1, 2, 3, 3, 2, 1), (0, 2, 1, 2, 0, 0), d = 138.$   
 $(1, 4, 5, 3, 3, 1) \rightarrow (1, 3, 3, 3, 3, 1), (0, 0, 2, 0, 0, 0), (0, 1, 0, 0, 0, 0), d = 130.$   
 $(1, 4, 5, 3, 3, 1) \rightarrow (1, 3, 3, 3, 3, 1), (0, 0, 1, 0, 0, 0), (0, 1, 1, 0, 0, 0), d = 135.$   
 $(1, 4, 5, 4, 2, 1) \rightarrow (1, 2, 3, 3, 2, 1), (0, 1, 2, 1, 0, 0), (0, 1, 0, 0, 0, 0), d = 123.$   
 $(1, 4, 5, 4, 2, 1) \rightarrow (1, 2, 3, 3, 2, 1), (0, 1, 1, 1, 0, 0), (0, 1, 1, 0, 0, 0), d = 128.$   
 $(1, 4, 5, 4, 2, 1) \rightarrow (1, 2, 2, 2, 2, 1), (0, 2, 3, 2, 0, 0), d = 137.$   
 $(1, 4, 6, 3, 2, 1) \rightarrow (1, 2, 2, 2, 2, 1), (0, 1, 4, 1, 0, 0), (0, 1, 0, 0, 0, 0), d = 122.$   
 $(1, 4, 6, 3, 2, 1) \rightarrow (1, 2, 2, 2, 2, 1), (0, 1, 3, 1, 0, 0), (0, 1, 1, 0, 0, 0), d = 126.$   
 $(1, 4, 6, 4, 1, 1) \rightarrow (1, 1, 1, 1, 1, 1), (0, 3, 5, 3, 0, 0), d = 132.$   
 $(1, 5, 3, 4, 3, 1) \rightarrow (1, 3, 3, 3, 3, 1), (0, 1, 0, 1, 0, 0), (0, 1, 0, 0, 0, 0), d = 142.$   
 $(1, 5, 4, 3, 3, 1) \rightarrow (1, 3, 3, 3, 3, 1), (0, 0, 1, 0, 0, 0), (0, 2, 0, 0, 0, 0), d = 129.$   
 $(1, 5, 4, 3, 3, 1) \rightarrow (1, 3, 3, 3, 3, 1), (0, 1, 1, 0, 0, 0), (0, 1, 0, 0, 0, 0), d = 135.$   
 $(1, 5, 4, 4, 2, 1) \rightarrow (1, 2, 3, 3, 2, 1), (0, 1, 1, 1, 0, 0), (0, 2, 0, 0, 0, 0), d = 122.$   
 $(1, 5, 4, 4, 2, 1) \rightarrow (1, 2, 3, 3, 2, 1), (0, 1, 0, 1, 0, 0), (0, 1, 1, 0, 0, 0), (0, 1, 0, 0, 0, 0), d = 128.$   
 $(1, 5, 4, 4, 2, 1) \rightarrow (1, 2, 2, 2, 2, 1), (0, 2, 2, 2, 0, 0), (0, 1, 0, 0, 0, 0), d = 133.$   
 $(1, 5, 4, 4, 2, 1) \rightarrow (1, 2, 2, 2, 2, 1), (0, 2, 1, 2, 0, 0), (0, 1, 1, 0, 0, 0), d = 139.$   
 $(1, 5, 4, 5, 1, 1) \rightarrow (1, 1, 1, 1, 1, 1), (0, 4, 3, 4, 0, 0), d = 124.$   
 $(1, 5, 5, 3, 2, 1) \rightarrow (1, 2, 3, 3, 2, 1), (0, 2, 2, 0, 0, 0), (0, 1, 0, 0, 0, 0), d = 120.$   
 $(1, 5, 5, 3, 2, 1) \rightarrow (1, 2, 2, 2, 2, 1), (0, 1, 3, 1, 0, 0), (0, 2, 0, 0, 0, 0), d = 121.$   
 $(1, 5, 5, 3, 2, 1) \rightarrow (1, 2, 2, 2, 2, 1), (0, 1, 2, 1, 0, 0), (0, 1, 1, 0, 0, 0), (0, 1, 0, 0, 0, 0), d = 126.$   
 $(1, 5, 5, 3, 2, 1) \rightarrow (1, 2, 2, 2, 2, 1), (0, 1, 1, 1, 0, 0), (0, 2, 2, 0, 0, 0), d = 135.$   
 $(1, 5, 5, 4, 1, 1) \rightarrow (1, 1, 1, 1, 1, 1), (0, 3, 4, 3, 0, 0), (0, 1, 0, 0, 0, 0), d = 128.$   
 $(1, 5, 5, 4, 1, 1) \rightarrow (1, 1, 1, 1, 1, 1), (0, 3, 3, 3, 0, 0), (0, 1, 1, 0, 0, 0), d = 133.$   
 $(1, 5, 6, 2, 2, 1) \rightarrow (1, 2, 2, 2, 2, 1), (0, 0, 1, 0, 0, 0), (0, 3, 3, 0, 0, 0), d = 126.$   
 $(1, 5, 6, 3, 1, 1) \rightarrow (1, 1, 1, 1, 1, 1), (0, 2, 3, 2, 0, 0), (0, 2, 2, 0, 0, 0), d = 129.$   
 $(1, 6, 3, 3, 3, 1) \rightarrow (1, 3, 3, 3, 3, 1), (0, 3, 0, 0, 0, 0), d = 127.$   
 $(1, 6, 3, 4, 2, 1) \rightarrow (1, 2, 3, 3, 2, 1), (0, 1, 0, 1, 0, 0), (0, 3, 0, 0, 0, 0), d = 120.$   
 $(1, 6, 3, 4, 2, 1) \rightarrow (1, 2, 2, 2, 2, 1), (0, 2, 1, 2, 0, 0), (0, 2, 0, 0, 0, 0), d = 127.$   
 $(1, 6, 4, 3, 2, 1) \rightarrow (1, 2, 3, 3, 2, 1), (0, 1, 1, 0, 0, 0), (0, 3, 0, 0, 0, 0), d = 113.$   
 $(1, 6, 4, 3, 2, 1) \rightarrow (1, 2, 2, 2, 2, 1), (0, 1, 2, 1, 0, 0), (0, 3, 0, 0, 0, 0), d = 119.$   
 $(1, 6, 4, 3, 2, 1) \rightarrow (1, 2, 2, 2, 2, 1), (0, 1, 1, 1, 0, 0), (0, 1, 1, 0, 0, 0), (0, 2, 0, 0, 0, 0), d = 124.$   
 $(1, 6, 4, 4, 1, 1) \rightarrow (1, 1, 1, 1, 1, 1), (0, 3, 3, 3, 0, 0), (0, 2, 0, 0, 0, 0), d = 122.$   
 $(1, 6, 4, 4, 1, 1) \rightarrow (1, 1, 1, 1, 1, 1), (0, 3, 2, 3, 0, 0), (0, 1, 1, 0, 0, 0), (0, 1, 0, 0, 0, 0), d = 127.$   
 $(1, 6, 5, 2, 2, 1) \rightarrow (1, 2, 2, 2, 2, 1), (0, 0, 1, 0, 0, 0), (0, 2, 2, 0, 0, 0), (0, 2, 0, 0, 0, 0), d = 116.$   
 $(1, 6, 5, 2, 2, 1) \rightarrow (1, 2, 2, 2, 2, 1), (0, 3, 3, 0, 0, 0), (0, 1, 0, 0, 0, 0), d = 126.$   
 $(1, 6, 5, 3, 1, 1) \rightarrow (1, 1, 1, 1, 1, 1), (0, 2, 3, 2, 0, 0), (0, 1, 1, 0, 0, 0), (0, 2, 0, 0, 0, 0), d = 119.$   
 $(1, 6, 5, 3, 1, 1) \rightarrow (1, 1, 1, 1, 1, 1), (0, 2, 2, 2, 0, 0), (0, 2, 2, 0, 0, 0), (0, 1, 0, 0, 0, 0), d = 127.$   
 $(1, 6, 6, 2, 1, 1) \rightarrow (1, 1, 1, 1, 1, 1), (0, 1, 2, 1, 0, 0), (0, 3, 3, 0, 0, 0), (0, 1, 0, 0, 0, 0), d = 120.$   
 $(1, 6, 6, 2, 1, 1) \rightarrow (1, 1, 1, 1, 1, 1), (0, 1, 1, 1, 0, 0), (0, 4, 4, 0, 0, 0), d = 135.$   
 $(1, 7, 3, 3, 2, 1) \rightarrow (1, 2, 3, 3, 2, 1), (0, 5, 0, 0, 0, 0), d = 107.$   
 $(1, 7, 3, 3, 2, 1) \rightarrow (1, 2, 2, 2, 2, 1), (0, 1, 1, 1, 0, 0), (0, 4, 0, 0, 0, 0), d = 115.$   
 $(1, 7, 3, 4, 1, 1) \rightarrow (1, 1, 1, 1, 1, 1), (0, 3, 2, 3, 0, 0), (0, 3, 0, 0, 0, 0), d = 114.$   
 $(1, 7, 4, 2, 2, 1) \rightarrow (1, 2, 2, 2, 2, 1), (0, 0, 1, 0, 0, 0), (0, 1, 1, 0, 0, 0), (0, 4, 0, 0, 0, 0), d = 108.$   
 $(1, 7, 4, 2, 2, 1) \rightarrow (1, 2, 2, 2, 2, 1), (0, 2, 2, 0, 0, 0), (0, 3, 0, 0, 0, 0), d = 114.$   
 $(1, 7, 4, 3, 1, 1) \rightarrow (1, 1, 1, 1, 1, 1), (0, 2, 3, 2, 0, 0), (0, 4, 0, 0, 0, 0), d = 111.$

$(1, 7, 4, 3, 1, 1) \rightarrow (1, 1, 1, 1, 1, 1), (0, 2, 2, 2, 0, 0), (0, 1, 1, 0, 0, 0), (0, 3, 0, 0, 0, 0), d = 115.$   
 $(1, 7, 5, 2, 1, 1) \rightarrow (1, 1, 1, 1, 1, 1), (0, 1, 2, 1, 0, 0), (0, 2, 2, 0, 0, 0), (0, 3, 0, 0, 0, 0), d = 109.$   
 $(1, 7, 5, 2, 1, 1) \rightarrow (1, 1, 1, 1, 1, 1), (0, 1, 1, 1, 0, 0), (0, 3, 3, 0, 0, 0), (0, 2, 0, 0, 0, 0), d = 119.$   
 $(1, 7, 6, 1, 1, 1) \rightarrow (1, 1, 1, 1, 1, 1), (0, 5, 5, 0, 0, 0), (0, 1, 0, 0, 0, 0), d = 128.$   
 $(1, 8, 3, 2, 2, 1) \rightarrow (1, 2, 2, 2, 2, 1), (0, 0, 1, 0, 0, 0), (0, 6, 0, 0, 0, 0), d = 101.$   
 $(1, 8, 3, 2, 2, 1) \rightarrow (1, 2, 2, 2, 2, 1), (0, 1, 1, 0, 0, 0), (0, 5, 0, 0, 0, 0), d = 104.$   
 $(1, 8, 3, 3, 1, 1) \rightarrow (1, 1, 1, 1, 1, 1), (0, 2, 2, 2, 0, 0), (0, 5, 0, 0, 0, 0), d = 105.$   
 $(1, 8, 4, 2, 1, 1) \rightarrow (1, 1, 1, 1, 1, 1), (0, 1, 2, 1, 0, 0), (0, 1, 1, 0, 0, 0), (0, 5, 0, 0, 0, 0), d = 100.$   
 $(1, 8, 4, 2, 1, 1) \rightarrow (1, 1, 1, 1, 1, 1), (0, 1, 1, 1, 0, 0), (0, 2, 2, 0, 0, 0), (0, 4, 0, 0, 0, 0), d = 106.$   
 $(1, 8, 5, 1, 1, 1) \rightarrow (1, 1, 1, 1, 1, 1), (0, 4, 4, 0, 0, 0), (0, 3, 0, 0, 0, 0), d = 111.$   
 $(1, 4, 3, 3, 3, 2, 1) \rightarrow (1, 2, 3, 3, 3, 2, 1), (0, 2, 0, 0, 0, 0, 0), d = 123.$   
 $(1, 4, 3, 3, 3, 2, 1) \rightarrow (1, 2, 2, 2, 2, 2, 1), (0, 1, 1, 1, 1, 0, 0), (0, 1, 0, 0, 0, 0, 0), d = 143.$   
 $(1, 4, 3, 4, 2, 2, 1) \rightarrow (1, 2, 2, 2, 2, 2, 1), (0, 0, 1, 1, 0, 0, 0), (0, 1, 0, 1, 0, 0, 0), (0, 1, 0, 0, 0, 0, 0), d = 131.$   
 $(1, 4, 3, 4, 2, 2, 1) \rightarrow (1, 2, 2, 2, 2, 2, 1), (0, 2, 1, 2, 0, 0, 0), d = 148.$   
 $(1, 4, 3, 4, 3, 1, 1) \rightarrow (1, 1, 1, 1, 1, 1, 1), (0, 2, 2, 2, 2, 0, 0), (0, 1, 0, 1, 0, 0, 0), d = 153.$   
 $(1, 4, 4, 3, 2, 2, 1) \rightarrow (1, 2, 2, 2, 2, 2, 1), (0, 0, 1, 1, 0, 0, 0), (0, 1, 1, 0, 0, 0, 0), (0, 1, 0, 0, 0, 0, 0), d = 124.$   
 $(1, 4, 4, 3, 2, 2, 1) \rightarrow (1, 2, 2, 2, 2, 2, 1), (0, 1, 2, 1, 0, 0, 0), (0, 1, 0, 0, 0, 0, 0), d = 131.$   
 $(1, 4, 4, 3, 2, 2, 1) \rightarrow (1, 2, 2, 2, 2, 2, 1), (0, 1, 1, 1, 0, 0, 0), (0, 1, 1, 0, 0, 0, 0), d = 137.$   
 $(1, 4, 4, 3, 3, 1, 1) \rightarrow (1, 1, 1, 1, 1, 1, 1), (0, 2, 2, 2, 2, 0, 0), (0, 0, 1, 0, 0, 0, 0), (0, 1, 0, 0, 0, 0, 0), d = 136.$   
 $(1, 4, 4, 3, 3, 1, 1) \rightarrow (1, 1, 1, 1, 1, 1, 1), (0, 2, 2, 2, 2, 0, 0), (0, 1, 1, 0, 0, 0, 0), d = 142.$   
 $(1, 4, 4, 4, 2, 1, 1) \rightarrow (1, 1, 1, 1, 1, 1, 1), (0, 1, 2, 2, 1, 0, 0), (0, 1, 1, 1, 0, 0, 0), (0, 1, 0, 0, 0, 0, 0), d = 126.$   
 $(1, 4, 4, 4, 2, 1, 1) \rightarrow (1, 1, 1, 1, 1, 1, 1), (0, 1, 2, 2, 1, 0, 0), (0, 1, 0, 1, 0, 0, 0), (0, 1, 1, 0, 0, 0, 0), d = 132.$   
 $(1, 4, 4, 4, 2, 1, 1) \rightarrow (1, 1, 1, 1, 1, 1, 1), (0, 1, 1, 1, 1, 0, 0), (0, 2, 2, 2, 0, 0, 0), d = 142.$   
 $(1, 4, 5, 2, 2, 2, 1) \rightarrow (1, 2, 2, 2, 2, 2, 1), (0, 0, 1, 0, 0, 0, 0), (0, 2, 2, 0, 0, 0, 0), d = 128.$   
 $(1, 4, 5, 3, 2, 1, 1) \rightarrow (1, 1, 1, 1, 1, 1, 1), (0, 1, 2, 2, 1, 0, 0), (0, 2, 2, 0, 0, 0, 0), d = 124.$   
 $(1, 4, 5, 3, 2, 1, 1) \rightarrow (1, 1, 1, 1, 1, 1, 1), (0, 1, 1, 1, 1, 0, 0), (0, 1, 3, 1, 0, 0, 0), (0, 1, 0, 0, 0, 0, 0), d = 126.$   
 $(1, 4, 5, 3, 2, 1, 1) \rightarrow (1, 1, 1, 1, 1, 1, 1), (0, 1, 1, 1, 1, 0, 0), (0, 1, 2, 1, 0, 0, 0), (0, 1, 1, 0, 0, 0, 0), d = 131.$   
 $(1, 4, 5, 4, 1, 1, 1) \rightarrow (1, 1, 1, 1, 1, 1, 1), (0, 3, 4, 3, 0, 0, 0), d = 136.$   
 $(1, 5, 3, 3, 2, 2, 1) \rightarrow (1, 2, 2, 2, 2, 2, 1), (0, 0, 1, 1, 0, 0, 0), (0, 3, 0, 0, 0, 0, 0), d = 120.$   
 $(1, 5, 3, 3, 2, 2, 1) \rightarrow (1, 2, 2, 2, 2, 2, 1), (0, 1, 1, 1, 0, 0, 0), (0, 2, 0, 0, 0, 0, 0), d = 130.$   
 $(1, 5, 3, 3, 3, 1, 1) \rightarrow (1, 1, 1, 1, 1, 1, 1), (0, 2, 2, 2, 2, 0, 0), (0, 2, 0, 0, 0, 0, 0), d = 135.$   
 $(1, 5, 3, 4, 2, 1, 1) \rightarrow (1, 1, 1, 1, 1, 1, 1), (0, 1, 2, 2, 1, 0, 0), (0, 1, 0, 1, 0, 0, 0), (0, 2, 0, 0, 0, 0, 0), d = 125.$   
 $(1, 5, 3, 4, 2, 1, 1) \rightarrow (1, 1, 1, 1, 1, 1, 1), (0, 1, 1, 1, 1, 0, 0), (0, 2, 1, 2, 0, 0, 0), (0, 1, 0, 0, 0, 0, 0), d = 138.$   
 $(1, 5, 4, 2, 2, 2, 1) \rightarrow (1, 2, 2, 2, 2, 2, 1), (0, 0, 1, 0, 0, 0, 0), (0, 1, 1, 0, 0, 0, 0), (0, 2, 0, 0, 0, 0, 0), d = 122.$   
 $(1, 5, 4, 2, 2, 2, 1) \rightarrow (1, 2, 2, 2, 2, 2, 1), (0, 2, 2, 0, 0, 0, 0), (0, 1, 0, 0, 0, 0, 0), d = 128.$   
 $(1, 5, 4, 3, 2, 1, 1) \rightarrow (1, 1, 1, 1, 1, 1, 1), (0, 1, 2, 2, 1, 0, 0), (0, 1, 1, 0, 0, 0, 0), (0, 2, 0, 0, 0, 0, 0), d = 118.$   
 $(1, 5, 4, 3, 2, 1, 1) \rightarrow (1, 1, 1, 1, 1, 1, 1), (0, 1, 1, 1, 1, 0, 0), (0, 1, 2, 1, 0, 0, 0), (0, 2, 0, 0, 0, 0, 0), d = 125.$   
 $(1, 5, 4, 3, 2, 1, 1) \rightarrow (1, 1, 1, 1, 1, 1, 1), (0, 1, 1, 1, 1, 0, 0), (0, 1, 1, 1, 0, 0, 0), (0, 1, 1, 0, 0, 0, 0),$   
 $(0, 1, 0, 0, 0, 0, 0), d = 131.$   
 $(1, 5, 4, 4, 1, 1, 1) \rightarrow (1, 1, 1, 1, 1, 1, 1), (0, 3, 3, 3, 0, 0, 0), (0, 1, 0, 0, 0, 0, 0), d = 132.$   
 $(1, 5, 4, 4, 1, 1, 1) \rightarrow (1, 1, 1, 1, 1, 1, 1), (0, 3, 2, 3, 0, 0, 0), (0, 1, 1, 0, 0, 0, 0), d = 138.$   
 $(1, 5, 5, 2, 2, 1, 1) \rightarrow (1, 1, 1, 1, 1, 1, 1), (0, 1, 1, 1, 1, 0, 0), (0, 0, 1, 0, 0, 0, 0), (0, 2, 2, 0, 0, 0, 0),$   
 $(0, 1, 0, 0, 0, 0, 0), d = 122.$   
 $(1, 5, 5, 2, 2, 1, 1) \rightarrow (1, 1, 1, 1, 1, 1, 1), (0, 1, 1, 1, 1, 0, 0), (0, 3, 3, 0, 0, 0, 0), d = 132.$   
 $(1, 5, 5, 3, 1, 1, 1) \rightarrow (1, 1, 1, 1, 1, 1, 1), (0, 2, 3, 2, 0, 0, 0), (0, 1, 1, 0, 0, 0, 0), (0, 1, 0, 0, 0, 0, 0), d = 125.$   
 $(1, 5, 5, 3, 1, 1, 1) \rightarrow (1, 1, 1, 1, 1, 1, 1), (0, 2, 2, 2, 0, 0, 0), (0, 2, 2, 0, 0, 0, 0), d = 134.$   
 $(1, 5, 6, 2, 1, 1, 1) \rightarrow (1, 1, 1, 1, 1, 1, 1), (0, 1, 2, 1, 0, 0, 0), (0, 3, 3, 0, 0, 0, 0), d = 125.$   
 $(1, 6, 3, 2, 2, 2, 1) \rightarrow (1, 2, 2, 2, 2, 2, 1), (0, 0, 1, 0, 0, 0, 0), (0, 4, 0, 0, 0, 0, 0), d = 117.$



- $d = 123$ .  
 $(1, 5, 3, 2, 2, 2, 1, 1) \rightarrow (1, 1, 1, 1, 1, 1, 1, 1), (0, 1, 1, 1, 1, 1, 0, 0), (0, 1, 1, 0, 0, 0, 0, 0), (0, 2, 0, 0, 0, 0, 0, 0),$   
 $d = 126$ .  
 $(1, 5, 3, 3, 2, 1, 1, 1) \rightarrow (1, 1, 1, 1, 1, 1, 1, 1), (0, 1, 2, 2, 1, 0, 0, 0), (0, 3, 0, 0, 0, 0, 0, 0), d = 118$ .  
 $(1, 5, 3, 3, 2, 1, 1, 1) \rightarrow (1, 1, 1, 1, 1, 1, 1, 1), (0, 1, 1, 1, 1, 0, 0, 0), (0, 1, 1, 1, 0, 0, 0, 0), (0, 2, 0, 0, 0, 0, 0, 0),$   
 $d = 128$ .  
 $(1, 5, 3, 4, 1, 1, 1, 1) \rightarrow (1, 1, 1, 1, 1, 1, 1, 1), (0, 3, 2, 3, 0, 0, 0, 0), (0, 1, 0, 0, 0, 0, 0, 0), d = 136$ .  
 $(1, 5, 4, 2, 2, 1, 1, 1) \rightarrow (1, 1, 1, 1, 1, 1, 1, 1), (0, 1, 1, 1, 1, 0, 0, 0), (0, 0, 1, 0, 0, 0, 0, 0), (0, 1, 1, 0, 0, 0, 0, 0),$   
 $(0, 2, 0, 0, 0, 0, 0, 0), d = 120$ .  
 $(1, 5, 4, 2, 2, 1, 1, 1) \rightarrow (1, 1, 1, 1, 1, 1, 1, 1), (0, 1, 1, 1, 1, 0, 0, 0), (0, 2, 2, 0, 0, 0, 0, 0), (0, 1, 0, 0, 0, 0, 0, 0),$   
 $d = 126$ .  
 $(1, 5, 4, 3, 1, 1, 1, 1) \rightarrow (1, 1, 1, 1, 1, 1, 1, 1), (0, 2, 3, 2, 0, 0, 0, 0), (0, 2, 0, 0, 0, 0, 0, 0), d = 123$ .  
 $(1, 5, 4, 3, 1, 1, 1, 1) \rightarrow (1, 1, 1, 1, 1, 1, 1, 1), (0, 2, 2, 2, 0, 0, 0, 0), (0, 1, 1, 0, 0, 0, 0, 0), (0, 1, 0, 0, 0, 0, 0, 0),$   
 $d = 129$ .  
 $(1, 5, 5, 2, 1, 1, 1, 1) \rightarrow (1, 1, 1, 1, 1, 1, 1, 1), (0, 1, 2, 1, 0, 0, 0, 0), (0, 2, 2, 0, 0, 0, 0, 0), (0, 1, 0, 0, 0, 0, 0, 0),$   
 $d = 120$ .  
 $(1, 5, 5, 2, 1, 1, 1, 1) \rightarrow (1, 1, 1, 1, 1, 1, 1, 1), (0, 1, 1, 1, 0, 0, 0, 0), (0, 3, 3, 0, 0, 0, 0, 0), d = 130$ .  
 $(1, 6, 2, 2, 2, 2, 1, 1) \rightarrow (1, 1, 1, 1, 1, 1, 1, 1), (0, 1, 1, 1, 1, 0, 0, 0), (0, 4, 0, 0, 0, 0, 0, 0), d = 120$ .  
 $(1, 6, 3, 2, 2, 1, 1, 1) \rightarrow (1, 1, 1, 1, 1, 1, 1, 1), (0, 1, 1, 1, 1, 0, 0, 0), (0, 0, 1, 0, 0, 0, 0, 0), (0, 4, 0, 0, 0, 0, 0, 0),$   
 $d = 115$ .  
 $(1, 6, 3, 2, 2, 1, 1, 1) \rightarrow (1, 1, 1, 1, 1, 1, 1, 1), (0, 1, 1, 1, 1, 0, 0, 0), (0, 1, 1, 0, 0, 0, 0, 0), (0, 3, 0, 0, 0, 0, 0, 0),$   
 $d = 118$ .  
 $(1, 6, 3, 3, 1, 1, 1, 1) \rightarrow (1, 1, 1, 1, 1, 1, 1, 1), (0, 2, 2, 2, 0, 0, 0, 0), (0, 3, 0, 0, 0, 0, 0, 0), d = 121$ .  
 $(1, 6, 4, 2, 1, 1, 1, 1) \rightarrow (1, 1, 1, 1, 1, 1, 1, 1), (0, 1, 2, 1, 0, 0, 0, 0), (0, 1, 1, 0, 0, 0, 0, 0), (0, 3, 0, 0, 0, 0, 0, 0),$   
 $d = 113$ .  
 $(1, 6, 4, 2, 1, 1, 1, 1) \rightarrow (1, 1, 1, 1, 1, 1, 1, 1), (0, 1, 1, 1, 0, 0, 0, 0), (0, 2, 2, 0, 0, 0, 0, 0), (0, 2, 0, 0, 0, 0, 0, 0),$   
 $d = 119$ .  
 $(1, 6, 5, 1, 1, 1, 1, 1) \rightarrow (1, 1, 1, 1, 1, 1, 1, 1), (0, 4, 4, 0, 0, 0, 0, 0), (0, 1, 0, 0, 0, 0, 0, 0), d = 124$ .  
 $(1, 7, 2, 2, 2, 1, 1, 1) \rightarrow (1, 1, 1, 1, 1, 1, 1, 1), (0, 1, 1, 1, 1, 0, 0, 0), (0, 5, 0, 0, 0, 0, 0, 0), d = 111$ .  
 $(1, 7, 3, 2, 1, 1, 1, 1) \rightarrow (1, 1, 1, 1, 1, 1, 1, 1), (0, 1, 2, 1, 0, 0, 0, 0), (0, 5, 0, 0, 0, 0, 0, 0), d = 107$ .  
 $(1, 7, 3, 2, 1, 1, 1, 1) \rightarrow (1, 1, 1, 1, 1, 1, 1, 1), (0, 1, 1, 1, 0, 0, 0, 0), (0, 1, 1, 0, 0, 0, 0, 0), (0, 4, 0, 0, 0, 0, 0, 0),$   
 $d = 110$ .  
 $(1, 7, 4, 1, 1, 1, 1, 1) \rightarrow (1, 1, 1, 1, 1, 1, 1, 1), (0, 3, 3, 0, 0, 0, 0, 0), (0, 3, 0, 0, 0, 0, 0, 0), d = 112$ .  
 $(1, 8, 2, 2, 1, 1, 1, 1) \rightarrow (1, 1, 1, 1, 1, 1, 1, 1), (0, 1, 1, 1, 0, 0, 0, 0), (0, 6, 0, 0, 0, 0, 0, 0), d = 102$ .  
 $(1, 8, 3, 1, 1, 1, 1, 1) \rightarrow (1, 1, 1, 1, 1, 1, 1, 1), (0, 2, 2, 0, 0, 0, 0, 0), (0, 5, 0, 0, 0, 0, 0, 0), d = 102$ .  
 $(1, 4, 2, 2, 2, 2, 1, 1) \rightarrow (1, 1, 1, 1, 1, 1, 1, 1), (0, 1, 1, 1, 1, 1, 0, 0), (0, 2, 0, 0, 0, 0, 0, 0), d = 133$ .  
 $(1, 4, 3, 2, 2, 2, 1, 1) \rightarrow (1, 1, 1, 1, 1, 1, 1, 1), (0, 1, 1, 1, 1, 1, 0, 0), (0, 0, 1, 0, 0, 0, 0, 0), (0, 2, 0, 0, 0, 0, 0, 0),$   
 $(0, 2, 0, 0, 0, 0, 0, 0), d = 127$ .  
 $(1, 4, 3, 2, 2, 2, 1, 1) \rightarrow (1, 1, 1, 1, 1, 1, 1, 1), (0, 1, 1, 1, 1, 1, 0, 0), (0, 1, 1, 0, 0, 0, 0, 0), (0, 1, 1, 0, 0, 0, 0, 0),$   
 $(0, 1, 0, 0, 0, 0, 0, 0), d = 130$ .  
 $(1, 4, 3, 3, 2, 1, 1, 1) \rightarrow (1, 1, 1, 1, 1, 1, 1, 1), (0, 1, 2, 2, 1, 0, 0, 0), (0, 2, 0, 0, 0, 0, 0, 0), d = 122$ .  
 $(1, 4, 3, 3, 2, 1, 1, 1) \rightarrow (1, 1, 1, 1, 1, 1, 1, 1), (0, 1, 1, 1, 1, 0, 0, 0), (0, 1, 1, 1, 0, 0, 0, 0), (0, 1, 1, 0, 0, 0, 0, 0),$   
 $(0, 1, 0, 0, 0, 0, 0, 0), d = 132$ .  
 $(1, 4, 3, 4, 1, 1, 1, 1) \rightarrow (1, 1, 1, 1, 1, 1, 1, 1), (0, 3, 2, 3, 0, 0, 0, 0), d = 144$ .  
 $(1, 4, 4, 2, 2, 1, 1, 1) \rightarrow (1, 1, 1, 1, 1, 1, 1, 1), (0, 1, 1, 1, 1, 0, 0, 0), (0, 0, 1, 0, 0, 0, 0, 0), (0, 1, 1, 0, 0, 0, 0, 0),$   
 $(0, 1, 1, 0, 0, 0, 0, 0), (0, 1, 0, 0, 0, 0, 0, 0), d = 124$ .  
 $(1, 4, 4, 2, 2, 1, 1, 1) \rightarrow (1, 1, 1, 1, 1, 1, 1, 1), (0, 1, 1, 1, 1, 0, 0, 0), (0, 2, 2, 0, 0, 0, 0, 0), d = 130$ .  
 $(1, 4, 4, 3, 1, 1, 1, 1) \rightarrow (1, 1, 1, 1, 1, 1, 1, 1), (0, 2, 3, 2, 0, 0, 0, 0), (0, 1, 0, 0, 0, 0, 0, 0), d = 127$ .





- $(1, 4, 2, 2, 2, 1, 1, 1, 1, 1, 1) \rightarrow (1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1), (0, 1, 1, 1, 1, 0, 0, 0, 0, 0, 0),$   
 $(0, 2, 0, 0, 0, 0, 0, 0, 0, 0, 0), d = 126.$
- $(1, 4, 3, 2, 1, 1, 1, 1, 1, 1, 1) \rightarrow (1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1), (0, 1, 2, 1, 0, 0, 0, 0, 0, 0, 0),$   
 $(0, 2, 0, 0, 0, 0, 0, 0, 0, 0, 0), d = 122.$
- $(1, 4, 3, 2, 1, 1, 1, 1, 1, 1, 1) \rightarrow (1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1), (0, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0),$   
 $(0, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0), (0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0), d = 125.$
- $(1, 4, 4, 1, 1, 1, 1, 1, 1, 1, 1) \rightarrow (1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1), (0, 3, 3, 0, 0, 0, 0, 0, 0, 0, 0), d = 127.$
- $(1, 5, 2, 2, 1, 1, 1, 1, 1, 1, 1) \rightarrow (1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1), (0, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0),$   
 $(0, 3, 0, 0, 0, 0, 0, 0, 0, 0, 0), d = 120.$
- $(1, 5, 3, 1, 1, 1, 1, 1, 1, 1, 1) \rightarrow (1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1), (0, 2, 2, 0, 0, 0, 0, 0, 0, 0, 0),$   
 $(0, 2, 0, 0, 0, 0, 0, 0, 0, 0, 0), d = 120.$
- $(1, 6, 2, 1, 1, 1, 1, 1, 1, 1, 1) \rightarrow (1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1), (0, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0),$   
 $(0, 4, 0, 0, 0, 0, 0, 0, 0, 0, 0), d = 114.$
- $(1, 7, 1, 1, 1, 1, 1, 1, 1, 1, 1) \rightarrow (1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1), (0, 6, 0, 0, 0, 0, 0, 0, 0, 0, 0), d = 108.$
- $(1, 4, 2, 2, 1, 1, 1, 1, 1, 1, 1) \rightarrow (1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1), (0, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0),$   
 $(0, 2, 0, 0, 0, 0, 0, 0, 0, 0, 0), d = 124.$
- $(1, 4, 3, 1, 1, 1, 1, 1, 1, 1, 1) \rightarrow (1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1), (0, 2, 2, 0, 0, 0, 0, 0, 0, 0, 0),$   
 $(0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0), d = 124.$
- $(1, 5, 2, 1, 1, 1, 1, 1, 1, 1, 1) \rightarrow (1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1), (0, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0),$   
 $(0, 3, 0, 0, 0, 0, 0, 0, 0, 0, 0), d = 119.$
- $(1, 6, 1, 1, 1, 1, 1, 1, 1, 1, 1) \rightarrow (1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1), (0, 5, 0, 0, 0, 0, 0, 0, 0, 0, 0), d = 114.$
- $(1, 4, 2, 1, 1, 1, 1, 1, 1, 1, 1) \rightarrow (1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1), (0, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0),$   
 $(0, 2, 0, 0, 0, 0, 0, 0, 0, 0, 0), d = 123.$
- $(1, 5, 1, 1, 1, 1, 1, 1, 1, 1, 1) \rightarrow (1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1), (0, 4, 0, 0, 0, 0, 0, 0, 0, 0, 0),$   
 $d = 119.$
- $(1, 4, 1, 1, 1, 1, 1, 1, 1, 1, 1) \rightarrow (1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1), (0, 3, 0, 0, 0, 0, 0, 0, 0, 0, 0),$   
 $d = 123.$

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